Assignment IV

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4 *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R*-modules.*

(a) *Show that if* C *is flat, then* A *is flat iff* B *is.*

Solution. Suppose that C and A are both flat. We follow the proof in T.Y. Lam, *Lectures on Modules and Rings* (Springer, 1998). First we prove the following lemma:

Lemma A right R-module is flat iff for any (finitely generated) left ideal $\mathfrak{a} \subseteq R$, the natural map $P \otimes_R \mathfrak{a} \to P\mathfrak{a}$ is an isomorphism of abelian groups. □

PROOF Since $P\mathfrak{a}$ is the image of the natural map from $P \otimes_R \mathfrak{a} \to P \otimes_R R$ if we identify $P \otimes_R R$ with P, the statement is equivalent to showing that $0 \to P \otimes_R \mathfrak{a} \to P \otimes_R R$ is exact. This is certainly true if P is flat. Conversely, if it is true for all finitely generated ideals $\mathfrak{a}$, then we get the same property for all left ideals by taking direct limits. Thus $\mathrm{Hom}_{\mathbb{Z}}(P \otimes_R R, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(P \otimes_R \mathfrak{a}, \mathbb{Q}/\mathbb{Z}) \to 0$ is exact and $\mathrm{Hom}_R(R, P') \to \mathrm{Hom}_R(\mathfrak{a}, P') \to 0$ is also exact, where P' is the character module of P. Thus by Baer's Criterion, P' is injective, so P is flat. ■

We have the following commutative diagram:

$$\begin{tikzcd}
A \otimes_R \mathfrak{a} \arrow[r, "\sigma"] \arrow[d, "\alpha"] & B \otimes_R \mathfrak{a} \arrow[r, "\tau"] \arrow[d, "\beta"] & C \otimes_R \mathfrak{a} \arrow[r] \arrow[d, "\gamma"] & 0 \\
A\mathfrak{a} \arrow[r, "\varphi"] & B\mathfrak{a} \arrow[r, "\psi"] & C\mathfrak{a} \arrow[r] & 0
\end{tikzcd}$$

Notice that α and γ are isomorphisms since A and C are flat. Now suppose $x \in \ker \beta$; then we see $0 = \psi(\beta(x)) = \gamma(\tau(x))$ so $x \in \ker \tau$. The top row is exact, so there is some $y \in A \otimes_R \mathfrak{a}$ so that $x = \sigma(y)$; this gives $0 = \beta(\sigma(y)) = \varphi(\alpha(y))$ so $y \in \ker \alpha$ (since φ is injective. Thus $y = 0$ and $x = \sigma(y) = 0$. So β is in fact an isomorphism, and thus B is flat.

Now assume that C and B are flat. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_R M \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \otimes_R M \arrow[r] \arrow[d, "\beta"] & C \otimes_R M \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \otimes_R N \arrow[r, "\psi"] & B \otimes_R N \arrow[r] & C \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

with a monomorphism $M \to N$. Since C is flat, by IV.5, both rows are exact (so φ and ψ are injective); then since C and B are flat, γ and β are injective. We need only show that α is injective. But since $\beta \circ \varphi$ is the composition of injective maps, it is injective; $\beta \circ \varphi = \psi \circ \alpha$ so since ψ is injective, α must be as well. ■

Solution 2. A shorter proof exists, courtesy of Professor Kearnes:

If C is flat, then the torsion long exact sequence associated to $0 \to A \to B \to C \to 0$ and a module M is

$$\cdots \to \mathrm{Tor}_n(A, M) \to \mathrm{Tor}_n(B, M) \to \mathrm{Tor}_n(C, M) \to \mathrm{Tor}_{n-1}(A, M) \to \mathrm{Tor}_{n-1}(B, M) \to \cdots$$

which, when C is flat, reduces to

$$\cdots \to 0 \to \mathrm{Tor}_n(A, M) \to 0 \to \mathrm{Tor}_{n-1}(A, M) \to \mathrm{Tor}_{n-1}(B, M) \to 0 \to \cdots$$

so $\mathrm{Tor}_n(A, M) \cong \mathrm{Tor}_n(B, M)$ for all M and n. Using the Tor characterization of flatness, A is flat iff these groups are 0 iff B is flat. ■

(b) *Give an example to show that the bi-implication in* **(a)** *may fail when C is not flat.*

Solution. Consider the short exact sequence of $\mathbb{Z}$ modules,

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus Z/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

as described in Dummit and Foote, *Abstract Algebra* (p 379). The module $\mathbb{Z}$ is flat, and the module $\mathbb{Z}/2\mathbb{Z}$ is not flat. It only remains to show that $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not flat. Since $\mathbb{Z} \oplus \mathbb{Z}/2$ is not torsion-free we can apply Proposition 3.1.4 and say that $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2, B) \neq 0$ for all abelian groups. However, by remark on page 67, $\mathrm{Tor}_n^{\mathbb{Z}}(B, \mathbb{Z} \oplus \mathbb{Z}/2) \neq 0$. Thus by Proposition 3.2.4, $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2, B) = 0$ is not flat. ∎