Problem 2. (Hower, Jones, Selker).

(a) Show that any $\mathbb{Z}_4$-module is isomorphic to one of the form $\left(\bigoplus^{\kappa}\mathbb{Z}_2\right)\oplus\left(\bigoplus^{\lambda}\mathbb{Z}_4\right)$ for some κ and λ.

(b) Explain how to determine the isomorphism type of $Tor_n^{\mathbb{Z}_4}(A,B)$ for any $\mathbb{Z}_4$-modules A and B.

Solution:

(a) Let $X \subseteq M$ be maximally independent over $\mathbb{Z}_4$. We know that X exists by Zorn's Lemma. Let $F = \langle X \rangle$ be the submodule generated by X. F is a free $\mathbb{Z}_4$-module because X is independent. In particular, we can write $F \cong \bigoplus^{\lambda}\mathbb{Z}_4$, where $\lambda = |X|$.

Let $M[2]$ be the submodule of M annihilated by $2 \in \mathbb{Z}_4$. Then $2F \leq M[2]$ since $2(2f) = 0$ for all $2f \in 2F$. Since $M[2]$ is annihilated by 2, the action of $\mathbb{Z}_2$ on M is faithful so we may view it as a $\mathbb{Z}_2$-module. Thus $M[2]$ is a vector space, so $2F$ has some complementing subspace, C, i.e. $M[2] = C \oplus 2F$. As letting κ be the dimension of C as a $\mathbb{Z}_2$-vector space, we see that $C \cong \bigoplus^{\kappa}\mathbb{Z}_2$. Note that, by definition of internal direct sum, we have $C \cap 2F = 0$.

We now claim that $M = C \oplus F$.

To see that $C \cap F = 0$, assume that $m \in C \cap F$. Then $m \in F$, so we can write $m = a_1x_1 + \cdots + a_nx_n$ for some $a_i \in \mathbb{Z}_4$, $x_i \in X$, $1 \leq i \leq n$. However, since $m \in C \subseteq M[2]$, we have $2m = 0 = 2a_1x_1 + \cdots + 2a_nx_n$. By independence of the x_i, we can conclude that $2a_i = 0$ for all i, so $a_i \in \{0, 2\}$ for all i. Therefore, $m \in 2F$. Also $m \in C$, so $m \in 2F \cap C = \{0\}$.

To see that $C + F = M$, suppose for a contradiction that $m \notin C + F$. Then in particular $m \notin X$, so m is dependent on some members of X. Thus we may write $am = \sum b_ix_i$ for some a different from 0. Now $m \notin C + F \supseteq M[2]$, so must have additive order four. Thus $am \neq 0$, and a cannot be a unit of $\mathbb{Z}_4$ (else $m \in F$), so $a = 2$. Thus $2am = 4m = 0$, so $2b_ix_i = 0$, so $b_i \in \{0, 2\}$ for all i. This shows that $2m = 2f$ for some $f \in F$. But now $2(m - f) = 0$, so $m - f \in M[2] \subseteq C + F$ and $f \in C + F$ so $m \in C + F$, contradiction.

Therefore $M = C \oplus F \cong \left(\bigoplus^{\lambda}\mathbb{Z}_2\right)\oplus\left(\bigoplus^{\kappa}\mathbb{Z}_4\right)$.

(b) Let A and B be $\mathbb{Z}_4$ modules. By part (a) we may write $A = \left(\bigoplus^{\kappa}\mathbb{Z}_2\right)\oplus\left(\bigoplus^{\lambda}\mathbb{Z}_4\right)$ for some κ and λ. Using the fact that Tor distributes over direct sum we have

$$\mathrm{Tor}_n(A,B) \cong \left(\bigoplus^{\kappa}\mathrm{Tor}_n(\mathbb{Z}_2,B)\right)\oplus\left(\bigoplus^{\lambda}\mathrm{Tor}_n(\mathbb{Z}_4,B)\right).$$

Let $B[2]$ be the submodule of B annihilated by 2. Then by calculation 3.1.6 on page 67 of [Weibel] we have that

$$\mathrm{Tor}_n(\mathbb{Z}_2,B) \cong \begin{cases} B/2B & \text{if } n = 0,\\ B[2]/2B & \text{if } n > 0 \end{cases} \quad\text{and}\quad \mathrm{Tor}_n(\mathbb{Z}_4,B) \cong \begin{cases} B & \text{if } n = 0,\\ \{0\} & \text{if } n > 0.\end{cases}$$

Thus

$$\mathrm{Tor}_n(A,B) \cong \begin{cases} \left(\bigoplus^{\kappa} B/2B\right) \oplus \left(\bigoplus^{\lambda} B\right) & \text{if } n = 0, \\ \bigoplus^{\lambda} B[2]/2B & \text{if } n > 0. \end{cases}$$

$\square$