Problem 1 (Selker, Pratarelli). *The torsion product of abelian groups A and B, written $\operatorname{Tor}(A,B)$, is presented by generators and relations with the generators being all triples $(a,m,b) \in A \times \mathbb{Z} \times B$ satisfying $ma = 0$ and $mb = 0$, and the relations all those of the form*

- $(a_1 + a_2, m, b) = (a_1, m, b) + (a_2, m, b)$, *provided* $ma_i = 0 = mb$,
- $(a, m, b_1 + b_2) = (a, m, b_1) + (a, m, b_2)$, *provided* $ma = 0 = mb_i$,
- $(a, mn, b) = (ma, n, b)$, *provided* $mna = 0 = nb$, *and*
- $(a, mn, b) = (a, m, nb)$, *provided* $ma = 0 = mnb$.

If A and B are finite, then $\operatorname{Tor}(A,B) \cong A \otimes B$, *but* $\operatorname{Tor}(\mathbb{Z},\mathbb{Z}) \not\cong \mathbb{Z} \otimes \mathbb{Z}$.

Proof. We claim:

$(*)$ If A and B are cyclic groups of prime power order, then $\operatorname{Tor}(A,B)$ is cyclic of order $\gcd(|A|, |B|)$.

$(**)$ $\operatorname{Tor}(A, B \oplus C) \cong \operatorname{Tor}(A,B) \oplus \operatorname{Tor}(A,C)$, and $\operatorname{Tor}(A \oplus B, C) \cong \operatorname{Tor}(A,C) \oplus \operatorname{Tor}(B,C)$.

For $(*)$, let $A = \langle 1_A \rangle$ and $B = \langle 1_B \rangle$ be groups of prime power orders p^α and q^β respectively. For brevity we will write $\underbrace{1_A + \ldots + 1_A}_{k-\text{times}}$ as $k1_A$ or just k when there is no chance of confusion. Note that $\operatorname{Tor}(A,B)$ consists of formal sums of elements of the form (a,m,b). Now if $p \neq q$ then for some $k, j \in \mathbb{Z}$ we have $kp^\alpha + jq^\beta = 1$. Thus for any $(a,m,b) \in \operatorname{Tor}(A,B)$ we have

$$\begin{aligned}
(a,m,b) &= (a, m, (kp^\alpha + jq^\beta)b) && (1)\\
&= (a, m, kp^\alpha b) + (a, m, jq^\beta b) && (2)\\
&= (a, kp^\alpha m, b) + (a, m, 0) && (3)\\
&= (kp^\alpha a, m, b) + (a, m, 0) && (4)\\
&= (0, m, b) + (a, m, 0). && (5)
\end{aligned}$$

Note that $(a,m,0) + (a,m,0) = (a,m,0)$, thus $(a,m,0) = 0$, and similarly for $(0,m,b)$. Thus the sum in (5) is equal to zero. Thus every generator is zero. Hence $\operatorname{Tor}(A,B)$ is the cyclic group of order one, as desired.

Now suppose that $q = p$. By symmetry, we may assume that $\alpha \leq \beta$. Consider the map defined on the generators of $\operatorname{Tor}(A,B)$ by $(a1_A, m, b1_B) \mapsto \left(\frac{m}{p^\beta} ab\right) 1_A$. It is straightforward to check that the relations are preserved, thus this partial map extends to a homomorphism $\operatorname{Tor}(A,B) \to \mathbb{Z}_{p^\alpha}$. Also, the element $g := (1_A, p^\beta, 1_B) \mapsto 1_A$, so g has order at least p^α. On the other hand,

$$p^\alpha g = p^\alpha(1_A, p^\beta, 1_B) = (p^\alpha 1_A, p^\beta, 1_B) = (0, p^\beta, 1_B) = 0$$

so g has order p^α, and we claim that g generates $\mathrm{Tor}(A,B)$. Let $(a,m,b) \in \mathrm{Tor}(A,B)$. Then $mb = 0$ so we may write $mb = kp^\beta$ for some $k \in \mathbb{Z}$. Then

$$(a,m,b) = (a,mb,1) = (a,kp^\beta,1) = (a,p^\beta,k) = k(a,p^\beta,1) = ka(1,p^\beta,1).$$

Thus every generator of $\mathrm{Tor}(A,B)$ has the form ng for some $n \in \mathbb{Z}$. Also,

$$ng + mg = n(1,p^\beta,1) + m(1,p^\beta,1) = (n,p^\beta,1) + (m,p^\beta,1) = (n+m,p^\beta,1) = (n+m)g,$$

so every element of $\mathrm{Tor}(A,B)$ is a multiple of g, and claim $(*)$ is proved.

For $(**)$ we define maps $\varphi_1 : \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,B \oplus C)$. and $\varphi_2 : \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,B \oplus C)$. Let $i_1 : B \hookrightarrow B \oplus C$ and $i_2 : C \hookrightarrow B \oplus C$ be the canonical inclusions. By the universal property of presentations the maps φ_1 and φ_2 are uniquely determined by where we send the generators. We define $\varphi_1 : (a,m,b) \mapsto (a,m,i_1(b))$ and $\varphi_2 : (a,m,c) \mapsto (a,m,i_2(c))$. Again, a straightforward calculation shows that the relations are preserved. Thus by the universal property of direct sum there exists a unique map $\varphi : \mathrm{Tor}(A,B) \oplus \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,B \oplus C)$ such that $\varphi \circ i_1 = \varphi_1$ and $\varphi \circ i_2 = \varphi_2$.

Now we construct an inverse for φ. We define $\varphi_1^{-1} : \mathrm{Tor}(A,B \oplus C) \to \mathrm{Tor}(A,B)$, by $\varphi_1^{-1} : (a,m,b \oplus c) \mapsto (a,m,b)$. Similarly, we define $\varphi_2^{-1} : \mathrm{Tor}(A,B \oplus C) \to \mathrm{Tor}(A,C)$ by $\varphi_2^{-1} : (a,m,b \oplus c) \mapsto (a,m,c)$. These partial maps extend to complete maps by the property of presentations. Then by the universal property of products we have a map $\varphi^{-1} : \mathrm{Tor}(A,B \oplus C) \to \mathrm{Tor}(A,B) \oplus \mathrm{Tor}(A,C)$ such that $\pi_1 \circ \varphi^{-1} = \varphi_1^{-1}$ and $\pi_2 \circ \varphi^{-1} = \varphi_2^{-1}$. Thus we have

$$\varphi \circ \varphi^{-1}((a,m,b \oplus c)) = \varphi_1 \oplus \varphi_2((a,m,b) \oplus (a,m,c)) = (a,m,b \oplus c),$$

so $\varphi \circ \varphi^{-1} = \mathrm{id}$. Similarly $\varphi^{-1} \circ \varphi = \mathrm{id}$, thus φ is an isomorphism, and $(**)$ is proved.

Using $(*)$ and $(**)$ we can classify (up to isomorphism) $\mathrm{Tor}(A,B)$ for any finite abelian groups A and B as follows. By the fundamental theorem of finitely generated abelian groups there are some finitely many primes, $p_0, \ldots, p_n$ and some natural numbers (some of which may be zero) $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ such that

$$A \cong \bigoplus_{i=0}^{n} \mathbb{Z}_{p_i^{\alpha_i}} \quad \text{and} \quad B \cong \bigoplus_{i=0}^{n} \mathbb{Z}_{p_i^{\beta_i}}.$$

Then, if $e_i := \min\{\alpha_i, \beta_i\}$, $(*)$ and $(**)$ yield $\mathrm{Tor}(A,B) \cong \bigoplus_{i=0}^{n} \mathbb{Z}_{p^{e_i}}$. The desired result follows because $(*)$ and $(**)$ also hold with $\mathrm{Tor}(A,B)$ replaced with $A \otimes B$ (see for example Dummit & Foote), so in the finite abelian case $\mathrm{Tor}(A,B) \cong A \otimes B$.

The isomorphism does not extend to infinite abelian groups. It is easy to see that $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, but $\mathrm{Tor}(\mathbb{Z},\mathbb{Z}) = \{0\}$ because $\mathbb{Z}$ is torsion-free. $\square$