Problem: Prove that homology commutes with direct sums: $H_n(\bigoplus C_i) \cong \bigoplus H_n(C_i)$.

Proof: Let $\{((C_i)_\bullet, d_i)\}_{i\in I}$ be a collection of chain complexes. Recall that the direct sum $\bigoplus C_i = \big((\bigoplus_{i\in I} C_i)_\bullet, d\big)$ is a chain complex with $d = \oplus_{i\in I} d_i$. That is every element $c \in \bigoplus C_i$ may be written $(c_i)_{i\in I}$ where $c_i \in C_i$ and all but finitely many of the c_i's are 0, denote this by (c_i). Then we have $dc = (d_i c_i)$. So $dc = 0$ if and only if $d_i c_i = 0$ for all $i \in I$. Thus we may define the maps

$$\phi : H_n(\bigoplus C_i) \to \bigoplus H_n(C_i) : [(c_i)] \mapsto ([c_i]),$$

and

$$\psi : \bigoplus H_n(C_i) \to H_n(\bigoplus C_i) : ([c_i]) \mapsto [(c_i)].$$

To see these are well defined note that $[(c_i)] = [(c_i')]$ if and only if $[0] = [(c_i - c_i')]$. This is true if and only if there is a $(b_i) \in \bigoplus C_i$ such that $d(b_i) = (d_i b_i) = (c_i - c_i')$. Which is true if and only if $c_i' + d_i b_i = c_i$ for each $i \in I$. That is to say $([c_i]) = ([c_i'])$ if and only if

$$\phi([(c_i)]) = ([c_i]) = ([c_i' + d_i b_i]) = ([c_i']) = \phi([(c_i')]).$$

Similarly we see that $([c_i]) = ([c_i'])$ if and only if $\psi(([c_i])) = \psi(([c_i']))$. Hence both ϕ and ψ are well defined. Clearly both are Abelian group homomorphisms. It is also clear that $\phi\psi([c_i]) = ([c_i])$, and $\psi\phi[(c_i)] = [(c_i)]$. Thus ϕ is an isomorphism with inverse ψ, and we have that $\bigoplus H_n(C_i) \cong H_n(\bigoplus C_i)$.