**Exercise 3.6**

*Prove that homology commutes with exact additive functors. That is, if $F : {}_R Mod \to {}_S Mod$ is an exact additive functor, then for every complex $C$ of $R$-modules and every $n \in \mathbb{Z}$ there is an $S$-module isomorphism $H_n(F(C)) \cong F(H_n(C))$.*

Suppose we are given the following complex $C$ of $R$-modules:

$$\cdots \quad C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

Applying $F$ to each component and differential of $C$ will give us the following sequence $F(C)$:

$$\cdots \; F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1}) \xrightarrow{F(d_{n-1})} \cdots$$

Because $F$ is additive, we have that $F(d_{n+1}) \circ F(d_n) = F(d_{n+1} \circ d_n) = F(0_{C_{n+1},C_{n-1}}) = 0$, and therefore $F(C)$ is a complex of $S$-modules.

The $n$th homology of $C$ and $F(C)$ respectively are expressed by the following short exact sequences:

$$0 \longrightarrow I_{n+1} \xrightarrow{\iota} K_n \xrightarrow{\nu} H_n(C) \longrightarrow 0$$

$$0 \longrightarrow I'_{n+1} \xrightarrow{\iota'} K'_n \xrightarrow{\nu'} H_n(F(C)) \longrightarrow 0$$

Here $I_{n+1}$ and $K_n$ are the image and kernel of $d_{n+1}$ and $d_n$ respectively, $I'_{n+1}$ and $K'_n$ are the image and kernel of $F(d_{n+1})$ and $F(d_n)$, and $\iota$ and $\nu$ are the usual inclusion and projection maps associated with a quotient. Because $F$ is exact, taking the image of the first of these short exact sequences gives another short exact sequence:

$$0 \longrightarrow F(I_{n+1}) \xrightarrow{F(\iota)} F(K_n) \xrightarrow{F(\nu)} F(H_n(C)) \longrightarrow 0$$

Suppose now that there exist isomorphisms $\alpha_{n+1}$ and $\beta_n$ such that the following diagram (*) holds:

$$\begin{tikzcd}
0 \arrow[r] & F(I_{n+1}) \arrow[r, "F(\iota)"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\circ"] & F(K_n) \arrow[r, "F(\nu)"] \arrow[d, "\beta_n"'] & F(H_n(C)) \arrow[r] & 0 \\
0 \arrow[r] & I'_{n+1} \arrow[r, "\iota'"] & K'_n \arrow[r, "\nu'"] & H_n(F(C)) \arrow[r] & 0
\end{tikzcd}$$

Then $0 = F(\nu) \circ F(\iota)$, hence $0 = F(\nu) \circ \beta^{-1} \circ \iota' \circ \alpha$ because the square commutes. Composing both sides with $\alpha^{-1}$ gives $0 = F(\nu) \circ \beta^{-1} \circ \iota'$. Now $\nu'$ is the cokernel of $\iota'$, hence the universal property of cokernels gives a map $\gamma : H_n(F(C)) \to F(H_n(C))$ where $\gamma \circ \nu' = F(\nu) \circ \beta^{-1}$. An identical argument gives a map

$\gamma' : F(H_n(C)) \to H_n(F(C))$, where $\gamma' \circ F(\nu) = \nu' \circ \beta$. Therefore $\gamma \circ \nu' \circ \beta = F(\nu)$, whence $\gamma' \circ \gamma \circ \nu' \circ \beta = \nu' \circ \beta$. The composition of two epimorphisms is epic, so $\gamma' \circ \gamma = \mathrm{id}$. Therefore the following diagram is commutative, and $\gamma$ is the desired isomorphism, provided $\alpha$ and $\beta$ exist.

$$\begin{tikzcd}
0 \arrow[r] & F(I_{n+1}) \arrow[r, "F(\iota)"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\circ"] & F(K_n) \arrow[r, "F(\nu)"] \arrow[d, "\beta_n"'] \arrow[dr, phantom, "\circ"] & F(H_n(C)) \arrow[r] \arrow[d, dashed, "\gamma"'] & 0 \\
0 \arrow[r] & I'_{n+1} \arrow[r, "\iota'"] & K'_n \arrow[r, "\nu'"] & H_n(F(C)) \arrow[r] & 0
\end{tikzcd}$$

We must now demonstrate the existence of (*). Consider the following diagram consisting of $S$-modules:

$$\begin{tikzcd}
F(C_n) \arrow[r, "F(d_n)"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\circ"] & F(C_{n-1}) \arrow[d, "\mathrm{id}"'] \\
F(C_n) \arrow[r, "F(d_n)"] & F(C_{n-1})
\end{tikzcd}$$

Now $d_n$ has an epi-mono factorization in ${}_R Mod$, given by $C_n \xrightarrow{\epsilon} I_n \xrightarrow{\mu} C_{n-1}$, and $F(d_n)$ has an epi-mono factorization in ${}_S Mod$, given by $F(C_n) \xrightarrow{\epsilon'} I'_n \xrightarrow{\mu'} F(C_{n-1})$. Note also that $F(d_n) = F(\mu \circ \epsilon) = F(\mu) \circ F(\epsilon)$, hence the image of the epi-mono factorization $C_n \xrightarrow{\epsilon} I_n \xrightarrow{\mu} C_{n-1}$ under $F$ is an epi-mono factorization of $F(d_n)$. By problem 2.5 we then have that the following diagram commutes, hence $\alpha_n$ is the desired isomorphism between $F(I_n)$ and $I'_n$.

$$\begin{tikzcd}
F(C_n) \arrow[r, "F(\epsilon)"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\circ"] & F(I_n) \arrow[r, "F(\mu)"] \arrow[d, shift right, "\alpha_n"'] \arrow[dr, phantom, "\circ"] & F(C_{n-1}) \arrow[d, "\mathrm{id}"'] \\
F(C_n) \arrow[r, "\epsilon'"] & I'_n \arrow[r, "\mu'"] \arrow[u, shift right, "\alpha'_n"'] & F(c_{n-1})
\end{tikzcd}$$

We use the commutativity of the first square to find an isomorphism $\beta_n : F(K_n) \to K'_n$. The map $d_n$ is expressible as $0 \to K_n \xrightarrow{\delta} C_n \xrightarrow{\epsilon} I_n \to 0$. We inject kernels on the left of the above diagram to produce the following diagram of short exact sequences. The proof of the existence of $\beta_n$ is the dual of that given earlier when proving the existence of $\gamma$, and is not repeated.

$$\begin{tikzcd}
0 \arrow[r] & F(K_n) \arrow[r, "F(\delta)"] \arrow[d, dashed, "\beta_n"'] \arrow[dr, phantom, "\circ"] & F(C_n) \arrow[r, "F(\epsilon)"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\circ"] & F(I_n) \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & K'_n \arrow[r, "\delta'"] & F(C_n) \arrow[r, "\epsilon'"] & I'_n \arrow[r] & 0
\end{tikzcd}$$

Having produced the desired isomorphisms, it remains to show that the following square commutes:

$$\begin{tikzcd}
F(I_{n+1}) \arrow[r, "F(\iota)"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\circ"] & F(K_n) \arrow[d, "\beta_n"'] \\
I'_{n+1} \arrow[r, "\iota'"] & K'_n
\end{tikzcd}$$

This is shown in the following diagram, in which all vertical rectangles commute.

$$\begin{tikzcd}
F(K_{n+1}) \arrow[rr] \arrow[dr] \arrow[dd, "\beta_{n+1}"'] & & F(C_{n+1}) \arrow[rr, "F(\epsilon)"] \arrow[dr, "F(d_{n+1})"] \arrow[dd, "\mathrm{id}"'] & & F(I_{n+1}) \arrow[dr] \arrow[dd, "\alpha_{n+1}"'] & \\
& F(K_n) \arrow[rr, "F(\delta)"] \arrow[dd, "\beta_n"'] & & F(C_n) \arrow[rr] \arrow[dd] & & F(I_n) \arrow[dd, "\alpha_n"'] \\
K'_{n+1} \arrow[rr] \arrow[dr] & & F(C_{n+1}) \arrow[rr, "\epsilon'"] \arrow[dr, "F(d_{n+1})"] & & I'_{n+1} \arrow[dr] & \\
& K'_n \arrow[rr, "\delta'"] & & F(C_n) \arrow[rr] & & I'_n
\end{tikzcd}$$

Comparing this diagram to (*) we see that $F(\iota) = F(\delta)^{-1} \circ F(d_{n+1}) \circ F(\epsilon)^{-1}$, and $\iota' = (\delta')^{-1} \circ F(d_{n+1}) \circ (\epsilon')^{-1}$, where these inverses are formed by choosing representatives. This process is well defined, because it is well defined back in ${}_R Mod$, and $F$ respects composition of morphisms. The commutativity of vertical rectangles in the above diagram then implies that $\beta_n \circ F(\iota) = \iota' \circ \alpha_{n+1}$, as desired. Hence the diagram (*) exists, and the proof is complete.