Problem 5. *Show that $C_\bullet$ is a split exact chain complex if and only if the identity map on $C_\bullet$ is null homotopic.*

Solution. Let $C_\bullet$ be a chain complex such that the identity $i_{C_\bullet}$ on $C_\bullet$ is null homotopic. Then $i_n = d_{n+1}s_n + s_{n-1}d_n$ for some chain homotopy $\{s_n\}$. Thus

$$d_n = d_n i_n = d_n(d_{n+1}s_n + s_{n-1}d_n) = d_n d_{n+1}s_n + d_n s_{n-1}d_n = d_n s_{n-1}d_n,$$

so $C_\bullet$ is split. Furthermore, since $i_{C_\bullet}$ is null homotopic, $(i_n)_* : H_n(C_\bullet) \to H_n(C_\bullet)$ is the zero map 0_*, so $H_n(C_\bullet) = 0$, thus $C_\bullet$ is acyclic.

Now suppose that $C_\bullet$ is split exact with splitting maps $\{s_n\}$. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & \\
 & & \operatorname{im}(d_{n+2}) \arrow[ur] \arrow[dr] & & & & \operatorname{im}(d_n) \arrow[ur] \arrow[dr] & & \\
\cdots \arrow[r] & C_{n+2} \arrow[ur] \arrow[rr, "d_{n+2}"] & & C_{n+1} \arrow[ll, bend left, "s_{n+1}"] \arrow[rr, "d_{n+1}"'] \arrow[dr] & & C_n \arrow[ll, bend right, "s_n"'] \arrow[ur] \arrow[rr, "d_n"] & & C_{n-1} \arrow[ll, bend left, "s_{n-1}"] \arrow[r] \arrow[dr] & \cdots \\
\iddots \arrow[ur] & & & & \operatorname{im}(d_{n+1}) \arrow[ur] \arrow[dr] & & & & \ddots \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

We see that since $C_\bullet$ is exact, then $\ker(d_n) = \operatorname{im}(d_{n-1})$, so each

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(d_{n+1}) \arrow[r] & C_n \arrow[r] & \operatorname{im}(d_n) \arrow[l, bend right, "s'_{n-1}"'] \arrow[r] & 0
\end{tikzcd}$$

is exact. Moreover, $s_{n-1} : C_{n-1} \to C_n$ can be restricted to a map $s'_{n-1} : \operatorname{im}(d_n) \to C_n$. We know that $d_n s_{n-1} d_n = d_n$, so $d_n s'_{n-1} d_n = d_n = i_{n-1} d_n$ and since d_n is surjective onto $\operatorname{im}(d_n)$, we have $d_n s'_{n-1} = i_{n-1}$, so the above short exact sequence splits. Then $C_n \cong \operatorname{im}(d_n) \oplus \operatorname{im}(d_{n+1})$. We see that each $d_n : C_n \to C_{n-1}$ is just projection onto the first coordinate, and then inclusion into the second coordinate: $d_n(a, a') = (0, a)$. Now define a chain homotopy $\{t_n\}$ by $t_n : C_n \to C_{n+1} : (a, a') \mapsto (a', 0)$. Then given any $(a, a') \in \operatorname{im}(d_n) \oplus \operatorname{im}(d_{n+1})$ we see that

$$(d_{n+1}t_n + t_{n-1}d_n)(a, a') = d_{n+1}(a', 0) + t_{n-1}(0, a) = (0, a') + (a, 0) = (a, a'),$$

so $d_{n+1}t_n + t_{n-1}d_n = i_n$, thus i_n is null homotopic.

$\square$