

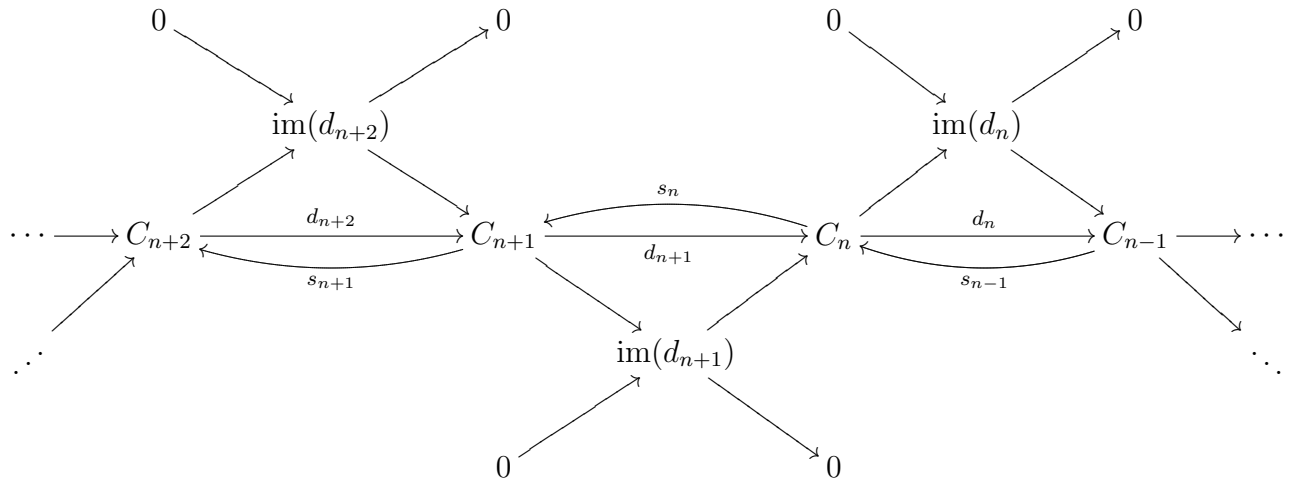
**Problem 5.** Show that  $C_\bullet$  is a split exact chain complex if and only if the identity map on  $C_\bullet$  is null homotopic.

*Solution.* Let  $C_\bullet$  be a chain complex such that the identity  $i_{C_\bullet}$  on  $C_\bullet$  is null homotopic. Then  $i_n = d_{n+1}s_n + s_{n-1}d_n$  for some chain homotopy  $\{s_n\}$ . Thus

$$d_n = d_n i_n = d_n(d_{n+1}s_n + s_{n-1}d_n) = d_n d_{n+1}s_n + d_n s_{n-1}d_n = d_n s_{n-1}d_n,$$

so  $C_\bullet$  is split. Furthermore, since  $i_{C_\bullet}$  is null homotopic,  $(i_n)_* : H_n(C_\bullet) \rightarrow H_n(C_\bullet)$  is the zero map  $0_*$ , so  $H_n(C_\bullet) = 0$ , thus  $C_\bullet$  is acyclic.

Now suppose that  $C_\bullet$  is split exact with splitting maps  $\{s_n\}$ . Consider the following diagram:



We see that since  $C_\bullet$  is exact, then  $\ker(d_n) = \text{im}(d_{n-1})$ , so each

$$0 \longrightarrow \text{im}(d_{n+1}) \longrightarrow C_n \xrightarrow{s'_{n-1}} \text{im}(d_n) \longrightarrow 0$$

is exact. Moreover,  $s_{n-1} : C_{n-1} \rightarrow C_n$  can be restricted to a map  $s'_{n-1} : \text{im}(d_n) \rightarrow C_n$ . We know that  $d_n s_{n-1} d_n = d_n$ , so  $d_n s'_{n-1} d_n = d_n = i_{n-1} d_n$  and since  $d_n$  is surjective onto  $\text{im}(d_n)$ , we have  $d_n s'_{n-1} = i_{n-1}$ , so the above short exact sequence splits. Then  $C_n \cong \text{im}(d_n) \oplus \text{im}(d_{n+1})$ . We see that each  $d_n : C_n \rightarrow C_{n-1}$  is just projection onto the first coordinate, and then inclusion into the second coordinate:  $d_n(a, a') = (0, a)$ . Now define a chain homotopy  $\{t_n\}$  by  $t_n : C_n \rightarrow C_{n+1} : (a, a') \mapsto (a', 0)$ . Then given any  $(a, a') \in \text{im}(d_n) \oplus \text{im}(d_{n+1})$  we see that

$$(d_{n+1}t_n + t_{n-1}d_n)(a, a') = d_{n+1}(a', 0) + t_{n-1}(0, a) = (0, a') + (a, 0) = (a, a'),$$

so  $d_{n+1}t_n + t_{n-1}d_n = i_n$ , thus  $i_n$  is null homotopic. □