Math 8174, Problem 4

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#4. Prove that the composition of two nullhomotopic chain maps is again nullhomotopic.

Proof. Let $(A_\bullet, d_\bullet), (B_\bullet, \delta_\bullet)$, and $(C_\bullet, \partial_\bullet)$ be chain complexes and let $f = \{f_n\}$ and $g = \{g_n\}$ be nullhomotopic chain maps from $A_\bullet$ to $B_\bullet$ and from $B_\bullet$ to $C_\bullet$ respectively. Then there exist chain homotopies $s : A_\bullet \to B_\bullet$ and $t : B_\bullet \to C_\bullet$ such that

$$f = sd + \delta s \text{ and } g = t\delta + \partial t.$$

Since g is a chain map, $g\delta = \partial g$ so that

$$\begin{aligned} gf &= g(sd + \delta s) \\ &= gsd + g\delta s \\ &= gsd + \partial gs \end{aligned}$$

Where gs, which consists of maps $g_{n+1}s_n : A_n \to C_{n+1}$, is a chain homotopy, hence gf is nullhomotopic. $\square$