HOMOLOGICAL ALGEBRA: HOMEWORK 3

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3)(a) Prove the Strong Four Lemma: if the diagram of modules

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\sigma"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r] & B' \arrow[r, "\tau"'] & C' \arrow[r] & D'
\end{tikzcd}$$

commutes, has exact rows, α is epic, and δ is monic, then $\ker\gamma = \sigma(\ker\beta)$ and $\operatorname{im}\beta = \tau^{-1}(\operatorname{im}\gamma)$.

(b) Use the Strong Four Lemma to prove the Five Lemma: if the diagram of modules

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

commutes and has exact rows then

(i) if α is epic, β and δ monic, then γ is monic.
(ii) if ε is monic, β and δ epic, then γ is epic.
(iii) if $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then so is γ.

SOLUTION

(a) *Proof.* Take $k \in \ker(\beta)$. By commutativity, $(\tau \circ \beta)(k) = (\gamma \circ \sigma)(k) = 0$, so $\sigma(k) \in \ker(\gamma)$; this shows that $\sigma(\ker(\beta)) \subseteq \ker(\gamma)$.

Conversely, take $k \in \ker(\gamma)$. Since $C \to C' \to D' = C \to D \to D'$, the latter sends $k \mapsto 0$. Since $\delta = D \to D'$ is monic we must have $k \in \ker(C \to D)$, and so by exactness there is some $l \in B$ with $\sigma(l) = k$. Now, $\gamma(k) = (\gamma\circ\sigma)(l) = (\tau\circ\beta)(l) = 0$, so $\beta(l) \in \ker(\tau) = \operatorname{im}(A' \to B')$. Since α is epic there is some $j \in A$ such that $A \to A' \to B'$ sends $j \mapsto \beta(l)$; let $i \in B$ be the image of j under $A \to B$. Then $j \in \ker\sigma$ and since $\beta(l) = \beta(j)$, we have $l - j \in \ker(\beta)$. Ergo, $\sigma(l - j) = k$ by choice of l. Thus $\ker(\gamma) \subseteq \sigma(\ker(\beta))$, and so the two sets are equal.

To see that $\operatorname{im}(\beta) = \tau^{-1}(\operatorname{im}(\gamma))$ note that for all $i \in B$ we have $(\tau \circ \beta)(i) = (\gamma \circ \sigma)(i)$, so $\operatorname{im}(\beta) \subseteq \tau^{-1}(\operatorname{im}(\gamma))$.

Conversely, take $i \in \tau^{-1}(\operatorname{im}(\gamma))$ and $j \in C$ such that $\tau(i) = \gamma(j)$. By exactness, $B' \to C' \to D'$ takes $i \mapsto 0$, so $\tau(i) \in \ker(C' \to D')$. Therefore $C \to C' \to D' = C \to D \to D'$ sends $j \mapsto 0$. Since $\delta = D \to D'$ is monic we must have $j \in \ker(C \to D) = \operatorname{im}(B \to C)$. Hence, there is some $l \in B$ with $\sigma(l) = j$. Now, $\tau(i) = \gamma(j) = (\gamma \circ \sigma)(l) = (\tau \circ \beta)(l)$, so $i - \beta(l) \in \ker(\tau) = \operatorname{im}(A' \to B')$. As α is epic, there some $k \in A$ such that $A \to A' \to B'$ takes $k \mapsto i - \beta(l)$; let $h \in B$ be the image of k under $A \to B$. Since $A \to B \to B' = A \to A' \to B'$, we have

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$\beta(h) = \beta(l) - i$, and hence $\beta(h - l) = i$. Ergo, $\tau^{-1}(\mathrm{im}(\gamma)) \subseteq \mathrm{im}(\beta)$, and so the two sets are equal. $\square$

(b) *Proof.* **(i)** Suppose that α is epic and β and δ are monic. Let $d_B : B \to C$ in the diagram. Then the Strong Four Lemma applied to the subdiagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

implies that $\ker\gamma = d_B(\ker\beta)$. Since β is monic, $\ker\beta = 0$. Thus $\ker\gamma = d_B(0) = 0$. Hence γ is monic.

(ii) Suppose that ε is monic and β and δ are epic. Let $d_{C'} : C' \to D'$ in the diagram. Then the Strong Four Lemma applied to the subdiagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

implies that $\mathrm{im}\,\gamma = d_{C'}^{-1}(\mathrm{im}\,\delta)$. Since δ is epic, $\mathrm{im}\,\delta = D'$. Thus $\mathrm{im}\,\gamma = d_{C'}^{-1}(D') = C'$. Hence γ is epic.

(iii) If $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then they are all both monic and epic. Therefore (i) and (ii) above imply that γ is both monic and epic and hence an isomorphism. $\square$