Math 8174, Assignment 3, Problem 2

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#2. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of complexes whose differentiations are all zero. Explicitly describe the connecting homomorphisms, explicitly describe the exact homology sequence, and verify the exactness of said sequence.

Proof. Recall the picture of relevant objects required in constructing the connecting homomorphism:

$$\begin{tikzcd}
 & \ker 0 \arrow[d, "\kappa_A"'] & \ker 0 \arrow[d, "\kappa_B"'] & \ker 0 \arrow[d, "\kappa_C"'] & \\
0 \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "0"] & B_n \arrow[r, "\beta_n"] \arrow[d, "0"] & C_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"'] \arrow[d, "\chi_A"] & B_{n-1} \arrow[r, "\beta_{n-1}"'] \arrow[d, "\chi_B"] & C_{n-1} \arrow[r] \arrow[d, "\chi_C"] & 0 \\
 & \operatorname{coker} 0 & \operatorname{coker} 0 & \operatorname{coker} 0 &
\end{tikzcd}$$

The connecting homomorphism should go from top right to bottom left; we showed it was well defined if we moved down along κ_C, "back" along an inverse for β_n, down along the $\mathcal{B}$ differential, "back" along an inverse for α_{n-1}, and down along χ_A. But moving down along the $\mathcal{B}$ differential will cause anything in B_n to go to zero. Since α_{n-1} is monic, the only thing it sends to zero is zero (i.e., the only morphisms that compose with α_{n-1} to be zero are zero morphisms). And of course χ_A will preserve a zero morphism. So the zero that arises in the middle of the connecting homomorphism persists to the end and causes the entire map to be zero.

Notice that the n-th homology group is the kernel of the n-th differentiation, modulo the image of the $n+1$-th differentiation. In this case the $n+1$-th differentiation is a zero map, so its image is zero, and the n-th differentiation is also a zero map, so its kernel is all of the nth object in the complex. Thus $H_n(\mathcal{A}) \cong A_n$, $H_n(\mathcal{B}) \cong B_n$, and so on. Thus our expected long exact sequence in homology

$$\cdots \xrightarrow{\alpha} H_{n+1}(\mathcal{B}) \xrightarrow{\beta} H_{n+1}(\mathcal{C}) \xrightarrow{\delta} H_n(\mathcal{A}) \xrightarrow{\alpha} H_n(\mathcal{B}) \xrightarrow{\beta} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A}) \xrightarrow{\alpha} \cdots$$

reduces to the sequence

$$\cdots \xrightarrow{\alpha} B_{n+1} \xrightarrow{\beta} C_{n+1} \xrightarrow{0} A_n \xrightarrow{\alpha} B_n \xrightarrow{\beta} C_n \xrightarrow{0} A_{n-1} \xrightarrow{\alpha} \cdots.$$

Recall that the maps between homology groups with the same subscript are given by restrictions of the maps between the objects of the complex with that subscript. (At least, this is true in a category of modules, which is sufficient for my purposes.) Exactness of this long sequence is guaranteed by the exactness of the original chain of complexes. At A_n, the image of the preceding map is trivial, and α is a monomorphism at all levels. Exactness at B_n is exactly a restatement of the exactness of the rows of the original diagram. At C_n, the image of the preceeding map is all of C_n since β is an epimorphism at all levels, and the following map is a zero map which has kernel C_n. So this sequence has been verifed to be exact.

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