

Math 8174, Assignment 2, Problem 7

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February 20, 2010

#7. Let \mathcal{A} be an abelian category. Given an object $A \in \mathcal{A}$, define an equivalence relation on pairs (X, α) , where $\alpha : X \rightarrow A$ is a morphism, by $(X, \alpha) \sim (X', \alpha')$ if there exists a $Y \in \mathcal{A}$ and epimorphisms β, β' such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & X \\ \beta' \downarrow & & \downarrow \alpha \\ X' & \xrightarrow{\alpha'} & A \end{array}$$

commutes. Prove that this does define an equivalence relation.

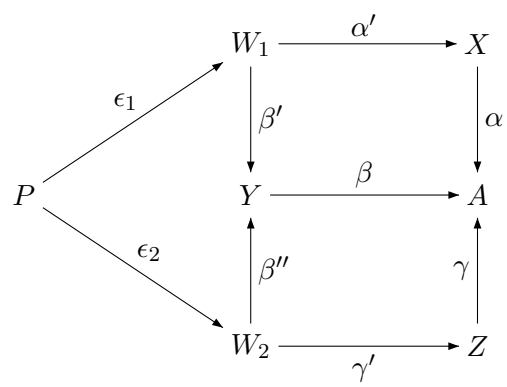
Proof. We must check reflexivity, symmetry, and transitivity. First, $(X, \alpha) \sim (X, \alpha)$ by the diagram

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ id_X \downarrow & & \downarrow \alpha \\ X & \xrightarrow{\alpha} & A \end{array}$$

(remember the identity map is both monic and epic). Second, suppose $(X, \alpha) \sim (X', \alpha')$, implying the diagram on the left below commutes. Then certainly the diagram on the right below commutes as well, giving $(X', \alpha') \sim (X, \alpha)$.

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & X \\ \beta' \downarrow & & \downarrow \alpha \\ X' & \xrightarrow{\alpha'} & A \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\beta'} & X' \\ \beta \downarrow & & \downarrow \alpha' \\ X & \xrightarrow{\alpha} & A \end{array}$$

Lastly, transitivity. Suppose $(X, \alpha) \sim (Y, \beta)$ and $(Y, \beta) \sim (Z, \gamma)$. That gives us a diagram as in the left “columns” of the diagram below (next page). Let P be the pullback of β' and β'' over Y . Then the full diagram as written below commutes by construction. We showed in class that epimorphisms can be pulled back (i.e., across a pullback square toward the pullback object) to epimorphisms. So $\alpha' \circ \epsilon_1$ and $\gamma' \circ \epsilon_2$ are epimorphisms, being the composition of epimorphisms. Then the object P and these epimorphisms demonstrate that (X, α) is related to (Z, γ) .



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