Math 8174, Assignment 2, Problem 7

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February 20, 2010

#7. Let $\mathcal{A}$ be an abelian category. Given an object $A \in \mathcal{A}$, define an equivalence relation on pairs (X, α), where $\alpha : X \to A$ is a morphism, by $(X, \alpha) \sim (X', \alpha')$ if there exists a $Y \in \mathcal{A}$ and epimorphisms β, β' such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "\beta'"'] & X \arrow[d, "\alpha"] \\ X' \arrow[r, "\alpha'"'] & A \end{tikzcd}$$

commutes. Prove that this does define an equivalence relation.

Proof. We must check reflexivity, symmetry, and transitivity. First, $(X, \alpha) \sim (X, \alpha)$ by the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[d, "\alpha"] \\ X \arrow[r, "\alpha"'] & A \end{tikzcd}$$

(remember the identity map is both monic and epic). Second, suppose $(X, \alpha) \sim (X', \alpha)$, implying the diagram on the left below commutes. Then certainly the diagram on the right below commutes as well, giving $(X', \alpha') \sim (X, \alpha)$.

$$\begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "\beta'"'] & X \arrow[d, "\alpha"] \\ X' \arrow[r, "\alpha'"'] & A \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\beta'"] \arrow[d, "\beta"'] & X' \arrow[d, "\alpha'"] \\ X \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Lastly, transitivity. Suppose $(X, \alpha) \sim (Y, \beta)$ and $(Y, \beta) \sim (Z, \gamma)$. That gives us a diagram as in the left "columns" of the diagram below (next page). Let P be the pullback of β' and β'' over Y. Then the full diagram as written below commutes by construction. We showed in class that epimorphisms can be pulled back (i.e., across a pullback square toward the pullback object) to epimorphisms. So $\alpha' \circ \epsilon_1$ and $\gamma' \circ \epsilon_2$ are epimorphisms, being the composition of epimorphisms. Then the object P and these epimorphisms demonstrate that (X, α) is related to (Z, γ).

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\begin{tikzcd}
 & W_1 \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & X \arrow[d, "\alpha"] \\
P \arrow[ur, "\epsilon_1"] \arrow[dr, "\epsilon_2"'] & Y \arrow[r, "\beta"] & A \\
 & W_2 \arrow[u, "\beta''"'] \arrow[r, "\gamma'"'] & Z \arrow[u, "\gamma"]
\end{tikzcd}
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$\square$