HOMOLOGICAL ALGEBRA: HOMEWORK II, PROBLEM 7

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(a) Prove that in any category with a zero object the projections of a product are epimorphisms and the coprojections of a coproduct are monomorphisms.
(b) Prove that in an abelian category a morphism is an isomorphism iff it is both a monomorphism and an epimorphism.

Solution

Proof. For the first part of (a), if $\{C_i : i \in I\}$ is a set of objects of $\mathcal{C}$, we consider their product $\prod_{i\in I} C_i$ and projection maps $\pi_j \colon \prod_{i\in I} C_i \to C_j (j\in I)$. To show all the projection maps are epimorphisms, we need to show that it is right cancellable.

$$\prod_{i\in I} C_i \xrightarrow{\pi_j} C_j \underset{g}{\overset{f}{\rightrightarrows}} X$$

Assume to the contrary: If we have two different morphisms: $f : C_j \to X$ and $g : C_j \to X$ (i.e. $f \neq g$), but $f\pi_j = g\pi_j$. Consider $\mathrm{id}_j \colon C_j \to C_j$ and $0_{ji} \colon C_j \to C_i$. The maps 0_{ij} together with id_j comprise a family of maps from C_j into the factors of $\prod_{i\in I} C_i$, hence induce a map $\alpha \colon C_j \to \prod_{i\in I} C_i$. So we have $f\pi_j\alpha = f\mathrm{id}_j = f \neq g = g\mathrm{id}_j = g\pi_j\alpha$. But this contradicts with $f\pi_j = g\pi_j$. Complete the proof of projections.

For the second part of (a), we have the similar result for coproducts followed by duality: Coprojections are monomorphisms.

For ($\Rightarrow$) direction of question (b), if $\theta : A \to B$ is an isomorphism, then there is a $\zeta : B \to A$ such that $\theta\zeta = id_B$ and $\zeta\theta = id_A$.

$$X \underset{g}{\overset{f}{\rightrightarrows}} A \underset{\zeta}{\overset{\theta}{\rightleftarrows}} B$$

If $f : X \to A$ and $g : X \to A$ are two morphisms such that $\theta f = \theta g$, then we left multiply ζ on both sides, we have $\zeta\theta f = \zeta\theta g$, so $id_A f = id_B g$, thus $f = g$. This shows that θ is left cancellable, so θ is a monomorphism.

$$A \underset{\zeta}{\overset{\theta}{\rightleftarrows}} B \underset{q}{\overset{p}{\rightrightarrows}} Y$$

Because of duality, we that θ is also an epimorphism.

For ($\Leftarrow$) direction, by Proposition 9.6 of [1], θ have a monic-epi factorization: $\theta = \nu\eta$,

$$A \overset{\eta}{\twoheadrightarrow} I \overset{\nu}{\rightarrowtail} B$$

with ν monic and η epi. If μ is the kernel of θ and ε is the cokernel of θ, we have

$$K \overset{\mu}{\rightarrowtail} A \overset{\eta}{\twoheadrightarrow} I \overset{\nu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$$

By Proposition 9.6, η is the cokernel of μ, but θ is also the cokernel of μ (because in abelian category every epimorphism is the cokernel of its kernel), so η and $\theta = \nu\eta$ are the same up to isomorphism: that means there is an isomorphism φ such that

$$\nu\eta = \theta = \varphi\eta$$

Because η is epi, we get $\nu = \varphi$ is an isomorphism.
Again because of duality, we can conclude that η is also an isomorphism.
So their composition $\theta = \nu\eta = \varphi\psi$ is also an isomorphism. Complete the proof. $\square$

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References

[1] Peter J. Hilton, Urs Stammbach, *A Course in Homological Algebra*, Graduate Texts in Mathematics, Vol.4, Springer-Verlag.