HOMOLOGICAL ALGEBRA: HOMEWORK 2

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5) Prove the following strong form of the remark made at the bottom of page 425.

(a) If $\mathcal{C}$ is an abelian category and $A \xrightarrow{\varphi} B$ is a morphism in $\mathcal{C}$, then φ can be factored as $A \xrightarrow{\varepsilon} I \xrightarrow{\mu} B$, where $\varepsilon = \operatorname{coker}(\ker(\varphi))$ and $\mu = \ker(\operatorname{coker}(\varphi))$.

(b) If

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A' \arrow[r, "\varphi'"'] & B'
\end{tikzcd} \tag{1}$$

commutes and φ' has an epi-mono factorization, then there is a unique morphism $I \xrightarrow{\gamma} I'$ such that

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & I \arrow[r, "\mu"] \arrow[d, dotted, "\gamma"'] & B \arrow[d, "\beta"] \\
A' \arrow[r, "\varepsilon'"'] & I' \arrow[r, "\mu'"'] & B'
\end{tikzcd} \tag{2}$$

commutes.

Solution

(a) *Proof.* We begin by proving the existence of such an ε. Since $\mu = \ker(\operatorname{coker}\varphi)$ and $(\operatorname{coker}\varphi)\varphi = 0$, φ must factor through μ via some unique morphism ε:

$$\begin{tikzcd}
& B \arrow[dr, "\operatorname{coker}\varphi"] & \\
& I \arrow[u, "\mu"] \arrow[r, "0"'] & Q \\
A \arrow[uur, bend left, "\varphi"] \arrow[ur, dotted, "\varepsilon"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

In order to show that ε is epi, we first prove a smaller claim:

Claim 1. *Under the assumptions of the problem, suppose that $A \xrightarrow{e} J \xrightarrow{m} B$ is a factorization of φ with m monic. Then there is unique monic $\alpha : I \to J$ such that $m\alpha = \mu$.*

Proof of Claim. $\mu = \ker(\operatorname{coker}\varphi)$, so

$$\operatorname{coker}\mu = \operatorname{coker}(\ker(\operatorname{coker}\varphi)) = \operatorname{coker}\varphi.$$

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Furthermore,

$$(\operatorname{coker} m)\varphi = (\operatorname{coker} m)me = 0e = 0.$$

Therefore $\operatorname{coker} m$ factors uniquely through $\operatorname{coker}\mu = \operatorname{coker}\varphi$ via a unique β such that the lower triangle in

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "e"'] & I \arrow[d, "\mu"] & \\
J \arrow[r, "m"'] & B \arrow[r, "\operatorname{coker} m"] \arrow[d, "\operatorname{coker}\mu"'] & C \\
& Q \arrow[ur, dotted, "\beta"'] &
\end{tikzcd}$$

commutes (the square commutes by assumption). This implies that

$$(\operatorname{coker} m)\mu = \beta(\operatorname{coker}\mu)\mu = \beta 0 = 0.$$

μ and m are monic, so $\ker(\operatorname{coker}\mu) = \mu$ and $\ker(\operatorname{coker} m) = m$. Thus μ factors through m. That is, there is unique $\alpha : I \to J$ such that $m\alpha = \mu$. Since μ is monic, α must also be monic. $\square$

Remark. In the dual of this claim, the maps $\mu^{op} = \operatorname{coker}(\ker(\varphi))$ and m^{op} are epic, the cokernels become kernels, and $\alpha^{op} : J \to I$ is epic.

Continuing with the proof of the problem, ε is epic if and only if $f\varepsilon = 0$ implies that $f = 0$. Suppose that $f\varepsilon = 0$ with $f : I \to C$. Then ε factors through the kernel of f, say $\varepsilon = (\ker f)\beta$ for some $\beta : A \to K$. Therefore $\varphi = \mu\varepsilon = \mu(\ker f)\beta$. Since $\mu(\ker f)$ is monic, from the claim above there is unique monic map $\alpha : I \to K$ such that $\mu(\ker f)\alpha = \mu$. μ is monic, so $(\ker f)\alpha = 1$. Therefore $(\ker f)\alpha(\ker f) = \ker f$, so $\alpha(\ker f) = 1$. Hence $\ker f$ and α are isomorphisms. But $\ker f$ is an isomorphism if and only if $f = 0$. Thus ε is epi.

Finally, we show that $\varepsilon = \operatorname{coker}(\ker\varphi)$. Let $e = \operatorname{coker}(\ker\varphi)$. Then since

$$\mu\varepsilon(\ker\varphi) = \varphi(\ker\varphi) = 0$$

and μ is monic, $\varepsilon(\ker\varphi) = 0$. Therefore there is unique β such that

$$\begin{tikzcd}
& K \arrow[d, "0"] \arrow[dr, "\ker\varphi"] \arrow[ddl, bend right, "0"'] & \\
& C \arrow[dl, dotted, "\beta"'] & A \arrow[l, "e"] \arrow[dll, bend left, "\varepsilon"] \\
I & &
\end{tikzcd}$$

commutes. Hence $\varphi = \mu\varepsilon = (\mu\beta)e$. Both ε and e are epi, so by the dual of the claim above, there is a unique epimorphism α such that $\alpha\varepsilon = e$. Therefore $\varepsilon = \beta e = \beta\alpha\varepsilon$. Since ε is epi, this implies that $\beta\alpha = 1$. Therefore $\alpha\beta\alpha = \alpha$, so $\alpha\beta = 1$. Hence α and β are isomorphisms, and ε and $e = \operatorname{coker}(\ker\varphi)$ are isomorphic. Thus $\varepsilon = \operatorname{coker}(\ker\varphi)$. $\square$

(b) *Proof.*

Claim 2. *Any two epi-mono factorizations of $\varphi : A \to B$ are isomorphic.*

Proof of Claim. It will be sufficient to show that any epi-mono factorization of φ is isomorphic to the factorization $\varphi = me$, where $e = \operatorname{coker}(\ker\varphi)$ and $m = \ker(\operatorname{coker}\varphi)$. Suppose that $\varphi = \varepsilon\mu$ is an epi-mono factorization of φ. Working from the previous claim and its dual, we have the following commutative diagram

$$\begin{tikzcd}
 & L \arrow[dl, "\overline{\kappa}"'] \arrow[d, "\ker\varepsilon"] & & \\
K \arrow[r, "\ker e"'] & A \arrow[r, "e"] \arrow[d, "\varepsilon"'] & J \arrow[d, "m"] \arrow[dl, dotted, bend left=10, "\theta"] & \\
 & I \arrow[r, "\mu"'] \arrow[ur, dotted, bend left=10, "\kappa"] & B \arrow[r, "\operatorname{coker}\mu"] \arrow[d, "\operatorname{coker}m"'] & R \\
 & & Q \arrow[ur, "\overline{\theta}"'] &
\end{tikzcd}$$

with κ epic and θ monic and unique such that the diagram commutes. The commutativity implies

$$me = \varphi = \mu\varepsilon = (m\kappa)(\theta e) = m(\kappa\theta)e$$

and

$$\mu\varepsilon = \varphi = me = (\mu\theta)(\kappa\varepsilon) = \mu(\theta\kappa)\varepsilon.$$

Since m is monic and e is epic, the first equality implies that $\kappa\theta = 1_J$. Similarly, since μ is monic and ε is epic, the second equality implies that $\theta\kappa = 1_I$. Hence κ is an isomorphism and ε is isomorphic to $e = \operatorname{coker}(\ker\varphi)$ and μ is isomorphic to $m = \ker(\operatorname{coker}\varphi)$. $\square$

Returning to the proof of the problem, by the above claim we may assume without loss of generality that $\varepsilon = \operatorname{coker}(\ker\varphi)$, $\mu = \ker(\operatorname{coker}\varphi)$, $\varepsilon' = \operatorname{coker}(\ker\varphi')$, and $\mu' = \ker(\operatorname{coker}\varphi')$. From the commutativity of the square in (1),

$$\mu'\varepsilon'\alpha(\ker\varphi) = \varphi'\alpha(\ker\varphi) = \beta\varphi(\ker\varphi) = \beta 0 = 0$$

and

$$(\operatorname{coker}\varphi')\beta\mu\varepsilon = (\operatorname{coker}\varphi')\beta\varphi = (\operatorname{coker}\varphi')\varphi'\alpha = 0\alpha = 0.$$

μ' is monic, so the first equality implies that $\varepsilon'\alpha(\ker\varphi) = 0$. ε is epi, so the second equality implies that $(\operatorname{coker}\varphi')\beta\mu = 0$. Since $\varepsilon = \operatorname{coker}(\ker\varphi)$ and $\mu' = \ker(\operatorname{coker}\varphi')$, there are unique $\gamma, \gamma' : I \to I'$ such that

$$\begin{tikzcd}
 & A \arrow[d, "\varepsilon"] \arrow[ddr, bend left, "\varepsilon'\alpha"] & \\
K \arrow[ur, "\ker\varphi"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & I \arrow[dr, dotted, "\gamma"] & \\
 & & I'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & B' \arrow[dr, "\operatorname{coker}\varphi'"] & \\
 & I' \arrow[u, "\mu'"] \arrow[r, "0"'] & Q \\
I \arrow[uur, bend left, "\beta\mu"] \arrow[ur, dotted, "\gamma'"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

commute. The commutativity of these diagrams and of the square in (1) yields

$$\mu'\gamma'\varepsilon = \beta\mu\varepsilon = \beta\varphi = \varphi'\alpha = \mu'\varepsilon'\alpha = \mu'\gamma\varepsilon.$$

Since μ' is monic, $\gamma'\varepsilon = \gamma\varepsilon$. Since ε is epi, $\gamma = \gamma'$. Finally, the commutativity of the two diagrams above immediately gives

$$\gamma\varepsilon = \varepsilon'\alpha \quad \text{and} \quad \mu'\gamma = \beta\mu,$$

proving that the rectangle (2) commutes. $\square$