Problem 4. *Let $\mathcal{C}$ be an abelian category. Show that if μ is monic and κ is epic then μ is a kernel of κ if and only if κ is a cokernel of μ.*

Solution. Let $\mathcal{C}$ be an abelian category, let $\mu : K \to A$ be monic and let $\kappa : A \to B$ be epic.

Suppose that $\kappa = \operatorname{coker} \mu$, then $\kappa \circ \mu = 0$. Since $\mathcal{C}$ is abelian and μ is monic we know that $\mu = \ker f$ for some $f : A \to C$ with $f \circ \mu = 0$. Then by the universal property of cokernels there exists a unique map $g : B \to C$ such that $g \circ \kappa = f$. Now let $h : X \to A$ be a map such that $\kappa \circ h = 0$. To show that $\mu = \ker \kappa$ we must find a unique $i : X \to K$ such that $\mu \circ i = h$. Now $f \circ h = g \circ \kappa \circ h = g \circ 0 = 0$, so since $\mu = \ker f$ there exists a unique $i : X \to K$ such that $\mu \circ i = h$. Then $\mu = \ker \kappa$. It follows by duality that if $\mu = \ker \kappa$, then $\kappa = \operatorname{coker} \mu$, thus $\mu = \ker \kappa \Leftrightarrow \kappa = \operatorname{coker} \mu$.

$$\begin{tikzcd}
K \arrow[rr, "\mu"] & & A \arrow[rr, "\kappa"] \arrow[dr, "f"'] & & B \arrow[dl, dotted, "\exists! g"] \\
& X \arrow[ul, dotted, "\exists! i"] \arrow[ur, "h"'] & & C &
\end{tikzcd}$$

$\square$