Problem: Imagine the integer polynomial ring in proper class of non-commuting variables:

$$\mathbb{Z}\left\langle \{x_\alpha | \alpha \text{ an ordinal}\}\right\rangle$$

Let $\mathfrak{C}$ be the category R-modules, by which we mean the category whose objects are ordinary Abelian groups equipped with one endomorphism for each x_α. The morphisms of $\mathfrak{C}$ are the Abelian group homomorphisms that respect the x_α's.

1. Show that $\mathfrak{C}$ is an Abelian category.
2. Show that $\mathfrak{C}$ is not equivalent to a category of modules over a true ring.

Proof: Let ${}_{\mathbb{Z}}\mathbf{Mod}$ denote the category of Abelian groups. For any two objects A and B in ${}_{\mathbb{Z}}\mathbf{Mod}$ we know $\mathrm{Hom}_{\mathbb{Z}}(A,B)$ is an Abelian group, furthermore we know that $\mathrm{Hom}_{\mathfrak{C}}(A,B) \subseteq \mathrm{Hom}_{\mathbb{Z}}(A,B)$. Clearly the zero map 0_{AB} preserves the action of R, so $\mathrm{Hom}_{\mathfrak{C}}(A,B)$ is nonempty. Let $f,g \in \mathrm{Hom}_{\mathfrak{C}}(A,B)$, for any $a \in A$ and any ordinal α, we have

$$(f-g)(x_\alpha a) = f(x_\alpha a) - g(x_\alpha a) = x_\alpha f(a) - x_\alpha g(a) = x_\alpha(f(a)-g(a)) = [x_\alpha(f-g)](a).$$

Thus $f - g \in \mathrm{Hom}_{\mathfrak{C}}(A,B)$, so by the subgroup criterion we have that $\mathrm{Hom}_{\mathfrak{C}}(A,B)$ is an Abelian group. Since $\mathrm{Hom}_{\mathfrak{C}}(A,B) \subseteq \mathrm{Hom}_{\mathbb{Z}}(A,B)$ we have that composition distributes over the sum in $\mathfrak{C}$.

As noted above all zero maps of ${}_{\mathbb{Z}}\mathbf{Mod}$ are in $\mathfrak{C}$. Since $x_\alpha 0 = 0$, for all α, we have that $0_{\mathbb{Z}} = 0_C$ is the zero object of $\mathfrak{C}$. Thus $\mathfrak{C}$ is a pre-additive category.

We know that in ${}_{\mathbb{Z}}\mathbf{Mod}$ the bi-product of A_1 and A_2 is just $(A_1 \oplus A_2, p_1, p_2, i_1, i_2)$ where $p_j : A_1 \oplus A_2 \to A_i : a_1 + a_2 \mapsto a_i$, and $i_j : A_j \to A_1 \oplus A_2 : a_j \mapsto a_j$ is the canonical inclusion into the j-th coordinate, for $j = 1,2$. All four of these maps clearly preserve the action by R. If B is an object in ${}_{\mathbb{Z}}\mathbf{Mod}$ along with maps $f_j : B \to A_i$, for $j = 1,2$, then we have the unique Abelian group homomorphism $\bar{f} : B \to A_1 \oplus A_2$ that makes $f_j = p_j \circ \bar{f}$ for $j = 1,2$ is given by $\bar{f}(b) = f_1(b) + f_2(b)$ for all $b \in B$. Thus if the f_j's are morphisms in $\mathfrak{C}$ for $i = 1,2$ then so is $\bar{f}$, since for every ordinal α we have

$$\bar{f}(x_\alpha b) = f_1(x_\alpha b) + f_2(x_\alpha b) = x_\alpha f_1(b) + x_\alpha f_2(b) = x_\alpha(f_1(b) + f_2(b)) = x_\alpha \bar{f}(b).$$

The uniqueness of $\bar{f}$ in ${}_{\mathbb{Z}}\mathbf{Mod}$ ensures the uniqueness of $\bar{f}$ in $\mathfrak{C}$ since all morphism in $\mathfrak{C}$ are Abelian group homomorphisms. Thus we have $(A_1 \oplus A_2, p_1, p_2)$ is a product in $\mathfrak{C}$. Nearly the same argument shows that it is also a coproduct in $\mathfrak{C}$. Thus $\mathfrak{C}$ has bi-products, making it is an additive category.

If $f \in \mathrm{Hom}_{\mathbb{Z}}(A,B)$ then we know that the kernel of f is given by (K,i) where $K = \{a \in A | f(a) = 0\}$ and $i : K \to A : a \mapsto a$. Clearly i preserves the R-action. If $g \in \mathrm{Hom}_{\mathbb{Z}}(C,A)$ such that $fg = 0$ then there is a unique map $G \in \mathrm{Hom}_{\mathbb{Z}}(C,K)$ such that $iG = g$. The map G is given by $G(c) = g(c)$. If g is a morphism in $\mathfrak{C}$ then so is G. Hence if $f \in \mathrm{Hom}_{\mathfrak{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathfrak{C}}(C,A)$ such that $fg = 0$ then we can take$K$ and $G \in \mathrm{Hom}_{\mathfrak{C}}(C,K)$ to be defined as above and we have $iG = g$. Again the uniqueness of G in $\mathfrak{C}$ is because G is unique in ${}_{\mathbb{Z}}\mathbf{Mod}$. Therefore (K, i) is the kernel of f in $\mathfrak{C}$, and the category $\mathfrak{C}$ has kernels.

Likewise nearly the same argument will show us that the cokernel $(B/\mathrm{im}(f), \nu)$, where ν is the canonical projection $\nu(b) = b + \mathrm{im}(f)$, is also the cokernel in $\mathfrak{C}$. Thus $\mathfrak{C}$ has cokernels.

Notice that since the kernels and cokernels in $\mathfrak{C}$ are the same as the kernels and cokernels in ${}_{\mathbb{Z}}\mathbf{Mod}$ we have that each monomorphism is the kernel of its cokernel, and each epimorphism is the cokernel of its kernel, since that is how it works in ${}_{\mathbb{Z}}\mathbf{Mod}$. Therefore we have that $\mathfrak{C}$ is an Abelian category.

Recall a few things about the category of S-modules for some ring S. Firstly there is a forgetful functor from ${}_S\mathbf{Mod}$ to the category $\mathfrak{S}$, of sets. Secondly this functor is representable. That is to say, as sets $M \cong \mathrm{Hom}_S(S, M)$ for every S-module M. Let us assume that $\mathfrak{C}$ is equivalent to the category ${}_S\mathbf{Mod}$ for some ring S. Then there is a forgetful functor $F : \mathfrak{C} \to \mathfrak{S}$. Also F is representable. Let Q be the object in $\mathfrak{C}$ that represents F, that is, for each object $M \in \mathfrak{C}$ there is a bijection of sets $\mathrm{Hom}_{\mathfrak{C}}(Q, M) \cong F(M) = M$. In particular as sets $Q \cong \mathrm{Hom}_{\mathfrak{C}}(Q, Q)$. Since this is a set there is an ordinal α such that for each $\beta > \alpha$ there exists a $\gamma \leq \alpha$ with $x_\beta = x_\gamma \in \mathrm{Hom}_{\mathfrak{C}}(Q, Q)$.

Let α and β be as above. Let $R_\beta = \mathbb{Z}\langle\{x_\gamma\}_{\gamma\leq\beta}\rangle$. Note that R_β is a free ring on the set β. Define the ring homomorphism $B : R_\beta \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$ by sending $x_\gamma = 1_{\mathbb{Z}}$ when $\gamma \leq \alpha$, $x_\gamma = 0_{\mathbb{Z}}$ when $\alpha < \gamma < \beta$, and $x_\beta = 2$ is left multiplication by 2. The map B gives $\mathbb{Z}$ an R_β module structure which we may extend to an R-module structure by setting $x_\gamma = 0$ for all $\gamma > \beta$. Thus we have an R-module structure on $\mathbb{Z}$, let us denote it $\mathbb{Z}_B$. By above we have that as sets $\emptyset \neq \mathbb{Z} \cong \mathrm{Hom}_{\mathfrak{C}}(Q, \mathbb{Z}_B)$. Thus there is a nonzero homomorphism $f : Q \to \mathbb{Z}_B$. Choose $a \in Q$ such that $f(a) \neq 0$. Then we have, for every $\gamma \leq \alpha$, that

$$f((x_\beta - x_\gamma)a) = (x_\beta - x_\gamma)f(a) = 2f(a) - f(a) = f(a) \neq 0.$$

Thus $x_\beta a - x_\gamma a \neq 0 \in Q$. Therefore we have that for every $\gamma \leq \alpha$ there is an $a \in Q$ such that $x_\beta a \neq x_\gamma a$. This implies that there exists a $\beta > \alpha$ such that for all $\gamma \leq \alpha$ we have $x_\beta \neq x_\gamma \in \mathrm{Hom}_{\mathfrak{C}}(Q)$. This is a contradiction to what we have stated above. Thus no such Q can exist, which implies that $\mathfrak{C}$ cannot be equivalent to a category of S-module for some ring S.