Topics In Algebra, Homework 2

Andrew Moorehead, Charlie Scherer

February 11, 2010

Problem (2)**.** *In the following every category is taken to be a full subcategory of,* ***Ab****, the category of abelian groups.*

a) The category of torsion abelian groups is an abelian category.

b) The category of torsion-free abelian groups is not an abelian category.

c) The category of finitely generated abelian groups is an abelian category.

d) The category of divisible groups is not an abelian category.

Proof. First we claim that if $\mathcal{A}$ is a (non-empty) full subcategory of **Ab** then $\mathcal{A}$ is an abelian category provided the following hold:

i) $\mathcal{A}$ is closed under direct products.

ii) If $G \in \mathrm{Ob}(\mathcal{A})$ then for every $H \leq G$ we have $H \in \mathrm{Ob}(\mathcal{A})$ and $G/H \in \mathrm{Ob}(\mathcal{A})$.

To see these are sufficient, suppose $\mathcal{A}$ is a full subcategory of **Ab** satisfying these two conditions. Take $G_1, G_2 \in \mathcal{A}$. Note that the zero subgroup of G_1 is a zero object in **Ab**. Indeed, all one element groups are isomorphic and since the canonical zero group in **Ab** has one element it is isomorphic to the zero subgroup of G_1. Thus, $\mathcal{A}$ contains a zero object from **Ab**. Now by hypothesis, $\mathcal{A}$ is closed with respect to the biproduct of **Ab**. Let $\phi : G_1 \to G_2$ be any element of $\mathrm{Hom}(G_1, G_2)$. We know that in **Ab**, $\ker(\phi)$ is the inclusion $\mathrm{Ker}(\phi) \hookrightarrow G_1$ where $\mathrm{Ker}(\phi)$ is the group theoretic kernel. Since $\mathrm{Ker}(\phi) \leq G_1$ it follows $\mathrm{Ker}(\phi) \in \mathrm{Ob}(\mathcal{A})$ and since $\mathcal{A}$ is full it follows $\ker(\phi) \in \mathrm{Mor}(\mathcal{A})$. Similarly, we know that in **Ab**, $\mathrm{cok}(\phi)$ is the natural projection $G_2 \twoheadrightarrow G_2/\phi[G_1]$. Since $\phi[G_1] \leq G_2$ we have $G_2/\phi[G_1] \in \mathrm{Ob}(\mathcal{A})$ and since $\mathcal{A}$ is full we have $\mathrm{cok}(\phi) \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed with respect to taking the zero object, biproducts, kernels and cokernels and is therefore an abelian category.

a) Let G and H be torsion abelian groups. Given $(g, h) \in G \times H$ there exists $n, m \in \mathbb{Z}^+$ such that $ng = 0$ and $mh = 0$. Since $nm \in \mathbb{Z}^+$ and $nm(g, h) = (0, 0)$ it follows that $G \times H$ is torsion. Thus, the direct product of torsion groups it torsion.

Now suppose that $H \leq G$. Then H is trivially torsion, for if $h \in H$ then by virtue of $h \in G$ there is some $n \in \mathbb{Z}^+$ such that $nh = 0$. We also have G/H torsion because given $g + H \in G/H$ there is some $n \in \mathbb{Z}^+$ such that $ng = 0$ and hence $n(g+H) = ng+H = H$. Ergo, a subgroup or quotient group of a torsion group is torsion and it follows the category of torsion abelian groups is an abelian category.

b) Let $\mathcal{I}$ be the full subcategory of torsion free abelian groups. Towards a contradition assume that $\mathcal{I}$ is abelian.

Consider the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ where $z \mapsto 2z$. First we claim this is a monomorphism in $\mathcal{I}$. Verily, as an element of **Ab** we know that ϕ is monic because it is injective. Since ϕ is left cancelable in **Ab** it is therefore left canelable $\mathcal{I}$. Thus, ϕ is monic in $\mathcal{I}$.

Since $\mathcal{I}$ is assumed abelian, let $\psi : \mathbb{Z} \to G$ be a cokernel of ϕ in $\mathcal{I}$.

$$\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\psi = \ker(\phi)} G$$

We claim ψ is equivalent to $0_{\mathbb{Z},0}$. Since $\psi \circ \phi = 0$ we must have $\psi(2z) = 0$ for all $z \in \mathbb{Z}$. Therefore, $2\mathbb{Z} \subset \mathrm{Ker}(\psi)$. As 2 is prime, $2\mathbb{Z}$ is a maximal ideal of the ring $\mathbb{Z}$, hence a maximal $\mathbb{Z}$-submodule of $\mathbb{Z}$. Thus, $\mathrm{Ker}(\psi) = 2\mathbb{Z}$ or $\mathrm{Ker}(\psi) = \mathbb{Z}$. If $\mathrm{Ker}(\psi) = 2\mathbb{Z}$ then $G \cong \mathbb{Z}/(2\mathbb{Z})$. But $\mathbb{Z}/(2\mathbb{Z})$ is not torsion free because $2(1+2\mathbb{Z}) = 2\mathbb{Z}$. Then we must have $\mathrm{Ker}(\psi) = \mathbb{Z}$ so that $G \cong \mathbb{Z}/\mathbb{Z} \cong 0$. Thus, the cokernel of ϕ is equivalent to $0_{\mathbb{Z},0}$.

But note ϕ is not the kernel of $\mathbb{Z} \to 0$. To see this suppose the contrary, namely that ϕ is a kernel of $0_{\mathbb{Z},0}$. Then since $0_{\mathbb{Z},0} \circ \mathrm{id}_{\mathbb{Z}} = 0_{\mathbb{Z},0}$ there must be some $\alpha : \mathbb{Z} \to \mathbb{Z}$ such that $\phi \circ \alpha = \mathrm{id}_{\mathbb{Z}}$. But $(\phi \circ \alpha)(1) = 2\alpha(1) \neq \mathrm{id}_{\mathbb{Z}}(1) = 1$ because 2 does not divide 1. Therefore, ϕ can not be the kernel of $0_{\mathbb{Z},0}$.

Now ϕ is monic that is not the kernel of its cokernel which is a very embarassing situation to have in an abelian cateogry.

c) Let G and H be finitely generated abelian groups with generating sets X and Y respectively . Then $(X \times \{1\}) \bigcup (\{1\} \times Y)$ is a generating set for $G \times H$. To see this suppose $g = \prod_{i=1}^{n} x_i$ and $h = \prod_{l=1}^{m} y_l$ for $g \in G$ and $h \in H$. Then $(g, h) = \prod_{i=1}^{n} (x_i, 1) \prod_{l=1}^{m} (1, y_l)$, hence the direct product of finitely generated abelian groups is a finitely generated abelian group.

Now suppose that $H \leq G$. A result of the classification of finitely generated abelian groups is that all $\mathbb{Z}$ modules are Noetherian. Therefore H is Noetherian, which is equivalent to being finitely generated. Furthermore, G/H is finitely generated, as the images of elements of X under the natural map generate the quotient. Therefore both subgroups and quotient groups of finitely generated abelian groups are abelian, hence the category of finitely generated abelian groups is an abelian category.

d) Let $\mathcal{D}$ be the full subcategory of divisible abelian groups. Assume that $\mathcal{D}$ is abelian.

In **Ab** we can form the following sequence of groups

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Q} \xrightarrow{\psi} \mathbb{Q}/\mathbb{Z}$$

where φ is the inclusion map and ψ is the natural map. Then φ and ψ are clearly kernel and cokernel with respect to each other in **Ab**. Both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, so $\mathbb{Q} \xrightarrow{\psi} \mathbb{Q}/\mathbb{Z}$ is a diagram in $\mathcal{D}$.

Since $\mathcal{D}$ is abelian, it has a kernel of ψ,

$$G \xrightarrow{\ker(\psi)} \mathbb{Q} \xrightarrow{\psi} \mathbb{Q}/\mathbb{Z}$$

Then G is the trivial group and $\ker(\psi) = 0_{0,\mathbb{Q}}$. To see this we work in **Ab**. By assumption $\psi \circ \ker(\psi) = 0$, hence $\ker(\psi)$ lifts uniquely to some $\lambda : G \to \mathbb{Z}$, where $\ker(\psi) = \varphi \circ \lambda$. $\ker(\psi)$ and φ are both monomorphisms, hence so is λ. Therefore λ embeds G into $\mathbb{Z}$. No subgroup of $\mathbb{Z}$ is divisible other than the trivial one, hence G is trivial.

By the dual of the argument in (b) we will find that ψ is not the cokernel of $\ker(\psi)$. Suppose contrarily that ψ is the cokernel of $\ker(\psi)$. Then $\mathrm{id}_{\mathbb{Q}} \circ \ker(\psi) = 0_{0,\mathbb{Q}}$, hence $\mathrm{id}_{\mathbb{Q}} = \alpha \circ \psi$ for a unique $\alpha : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$. However, $\alpha \circ \psi(1) = \alpha(0) = 0$, hence such an α cannot exist.

Therefore, ψ is not the cokernel of its kernel which contradicts the fact that it is epic.

$\square$