Problem 1 (Selker, Wakefield). ***(a)*** *$\mathbb{Z}_2$ is a cogroup in the category of abelian groups*
(b) *$\mathbb{Z}_2$ is not a cogroup in the category of all groups.*
(c) *$\mathbb{Z}_2$ is not a cogroup in the category of groups of exponent 4.*

Proof. **(a)** Because coproducts exist in the category of abelian groups we have that the functor $\mathrm{Hom}(G, \_)$ out of abelian groups lifts to a functor to groups iff G is a cogroup-object in the category of abelian groups. It is a well known fact (and routine to check) that if A and B are R-modules, then $\mathrm{Hom}(A, B)$ is an abelian group under pointwise addition of maps. Thus, as abelian groups are the same as $\mathbb{Z}$-modules, we have that $\mathrm{Hom}(\mathbb{Z}_2, A)$ is an abelian group whenever A is an abelian group. Moreover, if $f : A \to B$ is an abelian group homomorphism we have $f_* : \mathrm{Hom}(\mathbb{Z}_2, A) \to \mathrm{Hom}(\mathbb{Z}_2, B) : \alpha \mapsto f \circ \alpha$. Then $f_*(\alpha + \beta)(x) = f \circ (\alpha+\beta)(x) = f(\alpha(x)+\beta(x)) = f(\alpha(x))+f(\beta(x))$, so f_* is an abelian group homomorphism because f is. Thus $\mathrm{Hom}(\mathbb{Z}_2, \_)$ is a functor from abelian groups to groups, so $\mathbb{Z}_2$ must be a cogroup in the cagtegory of abelian groups.

(b) Suppose for a contradiction that $\mathbb{Z}_2$ were a cogroup in the category of all groups. We will observe how the possible comultiplication maps $\mu : \mathbb{Z}_2 \to \mathbb{Z}_2 \coprod \mathbb{Z}_2$ define the multiplication on $\mathrm{Hom}(\mathbb{Z}_2, S_3)$ to get a contradiction. Recall that $\mathbb{Z}_2 \coprod \mathbb{Z}_2$ is the free product on two (noncommuting) involutions, thus has a presentation $\mathbb{Z}_2 \coprod \mathbb{Z}_2 = \langle a, b : a^2 = b^2 = 1\rangle$. We will denote the generator of $\mathbb{Z}_2$ by x. Note that the only nonidentity elements of $\mathbb{Z}_2 \coprod \mathbb{Z}_2$ are words consisting of all "a"s and "b"s such that aa and bb do not occur (we assume now and henceforth that all words are reduced if possible). Note that if $w = w_1 \dots w_n$ where $w_i = a$ or b, then then $w^{-1} = w_n \dots w_1$. If a word w has even length then its first and last letter are different so that $w^{-1} \neq w$, and in particular $w^2 \neq 1$. Thus, as order two elements must map to something of order 1 or 2, we cannot have $\mu(x) = w$ for any even-length word w other than the identity. On the other hand if n is odd and $w = w_1 \dots w_n$ then $w_1 = w_n$, $w_2 = w_{n-1}$, and so on, so that $w^2 = 1$. Thus we may have $\mu(x) = w$ for any odd-length word $w \in \mathbb{Z}_2 \coprod \mathbb{Z}_2$ or $\mu(x) = 1$. We consider first the case $\mu(x) \neq 1$, and fix $w = \mu(x) = w_1 \dots w_n$. Then

$$\begin{aligned} w &= w_1 \dots w_{\frac{n-1}{2}-1} w_{\frac{n-1}{2}} w_{\frac{n-1}{2}+1} \dots w_n \\ &= w_1 \dots w_{\frac{n-1}{2}-1} w_{\frac{n-1}{2}} w_{\frac{n-1}{2}-1} \dots w_1 \\ &= w_1 \dots w_{\frac{n-1}{2}-1} w_{\frac{n-1}{2}} \left(w_1 \dots w_{\frac{n-1}{2}-1}\right)^{-1}. \end{aligned}$$

Thus for some u, we have either $w = uau^{-1}$ or $w = ubu^{-1}$. By symmetry of the argument, we may assume that $w = uau^{-1}$. We have inclusions, $i_1 : \mathbb{Z}_2 \to \mathbb{Z}_2 \coprod \mathbb{Z}_2 : x \mapsto a$, and $i_2 : x \mapsto b$. Then for any $\alpha, \beta \in \mathrm{Hom}(\mathbb{Z}_2, S_3)$, we have the following commutative diagram.

$$
\begin{tikzcd}
\mathbb{Z}_2 \arrow[r, "\mu"] & \mathbb{Z}_2 \coprod \mathbb{Z}_2 \arrow[dr, dotted, "\alpha \amalg \beta"] & \mathbb{Z}_2 \arrow[l, "i_1"'] \arrow[d, "\alpha"] \\
& \mathbb{Z}_2 \arrow[u, "i_2"] \arrow[r, "\beta"'] & S_3
\end{tikzcd}
$$

Multiplication on $\mathrm{Hom}(\mathbb{Z}_2, S_3)$ is thus given by $\alpha \star \beta = (\alpha \amalg \beta) \circ \mu$. Note that because $(\alpha \amalg \beta) \circ i_1 = \alpha$ we have $\alpha \amalg \beta(a) = \alpha \amalg \beta(i_1(x)) = \alpha(x)$. Letting $\alpha : x \mapsto e$ where e is the identity of S_3, we have,

$$
\begin{aligned}
\alpha \star \beta(x) &= (\alpha \amalg \beta) \circ \mu(x) \\
&= \alpha \amalg \beta(uau^{-1}) \\
&= \alpha \amalg \beta(u) \cdot \alpha \amalg \beta(a) \cdot \alpha \amalg \beta(u^{-1}) \\
&= \alpha \amalg \beta(u) \cdot \alpha(x) \cdot (\alpha \amalg \beta(u))^{-1} \\
&= \alpha \amalg \beta(u) \cdot e \cdot (\alpha \amalg \beta(u))^{-1} \\
&= e = \alpha(x).
\end{aligned}
$$

Thus β must be the identity of $\mathrm{Hom}(\mathbb{Z}_2, S_3)$. However, as β was arbitrary, this implies that $\mathrm{Hom}(\mathbb{Z}_2, S_3)$ is the one element group, a contradiction. On the other hand if $\mu(x) = 1$ were the trivial homomorphism then also any composition $f \circ \mu$ is trivial, so again $\mathrm{Hom}(\mathbb{Z}_2, S_3)$ would have one element, which it does not. So $\mathbb{Z}_2$ is not a cogroup in the category of all groups.

(c) Suppose again for a contradiction that $\mathbb{Z}_2$ were a cogroup in the category of groups of exponent four. We will again consider the possible comultiplication maps $\mu : \mathbb{Z}_2 \to \mathbb{Z}_2 \coprod \mathbb{Z}_2$, but the coproduct here is different, namely we have that $\mathbb{Z}_2 \coprod \mathbb{Z}_2$ is presented by $\langle a, b : a^2 = b^2 = (ab)^4 = (ba)^4 = 1 \rangle$. We will prove that $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \coprod \mathbb{Z}_2)$ is not a group. Again we may have $\mu(x) = 1$, $\mu(x) = uau^{-1}$ or $\mu(x) = ubu^{-1}$ for some word u. The only additional possibility is $\mu(x) = (ab)^2 = (ba)^2$. Multiplication $\star$ on $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \coprod \mathbb{Z}_2)$ is determined by μ as above. First note that if $\mu(x) = 1$ then for all $\alpha, \beta \in \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \coprod \mathbb{Z}_2)$ we would have $\alpha \star \beta = \alpha \amalg \beta \circ \mu = \mu$, so the group is trivial, a contradiction. Let us now consider the possibility that $\mu(x) = uau^{-1}$ for some u. Then if $\alpha : x \mapsto 1$ we will have for any β,

$$
\begin{aligned}
\alpha \star \beta(x) &= \alpha \amalg \beta \circ \mu(x) \\
&= \alpha \amalg \beta(uau^{-1}) \\
&= \alpha \amalg \beta(u)\alpha(x)(\alpha \amalg \beta(u))^{-1} \\
&= \alpha \amalg \beta(u)(\alpha \amalg \beta(u))^{-1} \\
&= 1 = \alpha(x),
\end{aligned}
$$

Which again would imply that $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \coprod \mathbb{Z}_2)$ is the trivial group, a contradiction. By symmetry, $\mu(x) = ubu^{-1}$ is also impossible. Now consider the case $\mu(x) = (ab)^2$. Again take $\alpha : x \mapsto 1$ and let β be arbitrary. Then

$$\begin{aligned}
\alpha \star \beta(x) &= \alpha \amalg \beta \circ \mu(x) \\
&= \alpha \amalg \beta(abab) \\
&= \alpha(x)\beta(x)\alpha(x)\beta(x) \\
&= (\beta(x))^2 \\
&= 1 = \alpha(x),
\end{aligned}$$

a contradiction as above. Thus $\mathbb{Z}_2$ cannot be a cogroup-object in the category of all groups of exponent four. $\square$