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Homework 1, Problem 8

Let $\mathcal{C} := \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$ be a small category and $\mathcal{S}$ the category of sets. As $\mathcal{C}$ is small, M is a set, so for each $A \in O$, $\mathrm{HomsTo}(A) := \{f \in M : \mathrm{cod}(f) = A\}$ is also a set. Hence, we may define a map $F : \mathcal{C} \to \mathcal{S}$ as follows:

$$\begin{aligned} O \ni A &\mapsto \mathrm{HomsTo}(A) \\ M \ni f &\mapsto f \circ \_\_ \end{aligned}$$

Note that composition is performed from right to left.

We first check that F is a functor. Note that for each $A, B \in O$ and $f \in \mathrm{Hom}(A, B)$, $F(f)$ maps from $\mathrm{HomsTo}(A)$ to $\mathrm{HomsTo}(B)$, so F preserves dom and cod. Next, $F(\mathrm{id}_A)$ is the map from $\mathrm{HomsTo}(A)$ to $\mathrm{HomsTo}(A)$ sending each $\varphi \in \mathrm{HomsTo}(A)$ to $\varphi \circ \mathrm{id}_A = \varphi$, so F preserves id. Finally we check composition which follows easily from the associative law for a category. Choose $A, B, C \in O$ with $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$. Then, $F(f) \in \mathrm{Hom}(\mathrm{HomsTo}(A), \mathrm{HomsTo}(B))$ and $F(g) \in \mathrm{Hom}(\mathrm{HomsTo}(B), \mathrm{HomsTo}(C))$. For each $\varphi \in \mathrm{HomsTo}(A)$,

$$F(g \circ f)(\varphi) = (g \circ f) \circ \varphi = g \circ (f \circ \varphi) = F(g)(F(f)(\varphi)) = (F(g) \circ F(f))(\varphi).$$

We now show that F is an embedding. Suppose that $F(A) = F(B)$. Then $\mathrm{HomsTo}(A) = \mathrm{HomsTo}(B)$, so in particular $\mathrm{id}_A \in \mathrm{HomsTo}(B)$. Thus, $\mathrm{cod}(\mathrm{id}_A) = B$; of course we also have that $\mathrm{cod}(\mathrm{id}_A) = A$. We conclude that $A = B$, so F is 1-to-1 on objects. Now suppose that $F(f) = F(g)$ for $f, g \in M$. f and g must have the same domain and codomain, so assume that $f, g \in \mathrm{Hom}(A, B)$. As $F(f) = F(g)$, $f \circ \_\_ = g \circ \_\_$, so $f \circ \mathrm{id}_A = g \circ \mathrm{id}_A$. This implies that $f = g$, so F is 1-to-1 on morphisms. Therefore, F is an embedding.