Topics In Algebra, Homework 1

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Problem (7)**.** *Let O be the ordered pair functor from* ***Set*** *to itself ($O(A) = A \times A, O(f) = f \times f$). Let U be the unordered pair from* ***Set*** *to itself, ($U(A) = \{\{x, y\} : x, y \in A\}, U(f)(\{x, y\}) = \{f(x), f(y)\}$). Then there are exactly three natural transformations from O to U. They are given by*

1. $\eta_A : O(A) \to U(A)$ *where* $(x, y) \mapsto \{x, y\}$.
2. $\tau_A : O(A) \to U(A)$ *where* $(x, y) \mapsto \{x\}$.
3. $\theta_A : O(A) \to U(A)$ *where* $(x, y) \mapsto \{y\}$.

Proof. First we check these are indeed natural transformations. This amounts to checking that for any sets A, B and any function $f : A \to B$ the diagrams below commute (subscripts have been suppressed).

$$\begin{tikzcd} O(A) \arrow[r, "O(f)"] \arrow[d, "{\eta,\tau,\theta}"'] & O(B) \arrow[d, "{\eta,\tau,\theta}"] \\ U(A) \arrow[r, "U(f)"] & U(B) \end{tikzcd}$$

For all $x, y \in A$ we have the following three equalities:

1. $(U(f) \circ \eta_A)(x, y) = U(f)(\{x, y\}) = \{f(x), f(y)\} = \eta_B(f(x), f(y)) = (\eta_B \circ O(f))(x, y)$.
2. $(U(f) \circ \tau_A)(x, y) = U(f)(\{x\}) = \{f(x)\} = \tau_B(f(x), f(y)) = (\tau_B \circ O(f))(x, y)$.
3. $(U(f) \circ \theta_A)(x, y) = U(f)(\{y\}) = \{f(y)\} = \theta_B(f(x), f(y)) = (\theta_B \circ O(f))(x, y)$.

Thus η, τ and θ do define natural transformations from O to U.

To show that these are the only such, we offer two arguments. The first argument uses the Yoneda lemma (see, for example, MacLane's book on category theory). The second is self-contained, but is longer.

Argument 1. Recall from lecture that the functor O is naturally isomorphic to $\mathrm{Hom}(2, \_)$, where 2 is the set $\{0, 1\}$. By the Yoneda lemma the set of natural transformations from $\mathrm{Hom}(2, \_)$ to U are in bijective correspondence with the members of $U(2) = \{\{0, 1\}, \{0\}, \{1\}\}$, thus there are exactly three such transformations, so they must be the above.

Argument 2. Let ρ be any natural transformation from O to U and fix a set A. As an initial step, we want to show that ρ_A is necessarily one of η_A, τ_A or θ_A. Note that if A is empty then so are $O(A)$ and $U(A)$ so that there is a unique function $\rho_A : O(A) \to U(A)$, and necessarily, $\rho_A = \eta_A = \tau_A = \theta_A$. If $A = \{a\}$ is a singleton then $O(A) = \{(a, a)\}$ and $U(A) = A$, so there is once again a unique function $\rho_A : O(A) \to U(A)$, and again $\rho_A = \eta_A = \tau_A = \theta_A$. We may therefore assume that $|A| \geq 2$.

Let x and y be any (not necessarily distinct) elements of A. We claim that $\rho_A(x, y) \subseteq \{x, y\}$. To see this, let $f : A \to A$ be defined by

$$f(c) = \begin{cases} x & \text{if } c = x, \\ y & \text{otherwise.} \end{cases}$$

Note that $O(f)(x,y) = (x,y)$, and for arbitrary $a, b \in A$, $U(f)(\{a,b\}) \subseteq \{x,y\}$, so in particular we have $U(f)(\rho_A(x,y)) \subseteq \{x,y\}$. Now, by assumption that ρ is natural,

$$(\rho_A \circ O(f))(x,y) = \rho_A(x,y) = (U(f) \circ \rho_A)(x,y) \subseteq \{x,y\},$$

So the claim is verified.

Since $\emptyset \notin U(A)$ it follows that ρ_A agrees with one of η_A, τ_A or θ_A on (x,y). This is true for every pair, so in particular for any $a \in A$ we have $\rho_A(a,a) = \eta_A(a,a) = \tau_A(a,a) = \theta_A(a,a) = \{a\}$. Now suppose that $x \neq y$. For each pair of distinct elements $a, b \in A$, define $g_{(a,b)} : A \to A$ by

$$g_{(a,b)}(c) = \begin{cases} a & \text{if } c = x, \\ b & \text{otherwise.} \end{cases}$$

Note that as $x \neq y$ we have $O(g)(x,y) = (a,b)$. Also, by assumption that ρ is a natural transformation, we have the following diagram.

$$\begin{tikzcd}
O(A) \arrow[r, "O(g_{(a,b)})"] \arrow[d, "\rho_A"'] \arrow[dr, phantom, "\bigcirc"] & O(A) \arrow[d, "\rho_A"] \\
U(A) \arrow[r, "U(g_{(a,b)})"] & U(A)
\end{tikzcd}$$

We now consider three cases.

Case 1: $\rho_A(x,y) = \eta_A(x,y)$. Then for each $a, b \in A$ with $a \neq b$, we have $U(g_{(a,b)})\rho_A(x,y) = U(g_{(a,b)})(\{x,y\}) = \{a,b\}$. By commutativity of the above diagram,

$$(U(g_{(a,b)}) \circ \rho_A)(x,y) = U(g_{(a,b)})(\{x,y\}) = \{a,b\} = (\rho_A \circ O(g_{(a,b)}))(x,y) = \rho_A(a,b).$$

Therefore $\rho_A(a,b) = \eta_A(a,b)$ also for all pairs with $a \neq b$, so $\rho_A = \eta_A$.

Case 2: $\rho_A(x,y) = \tau_A(x,y) = \{x\}$. Then for all $a \neq b$, $U(g_{(a,b)})\rho_A(x,y) = U(g_{(a,b)})(\{x\}) = \{a\}$, therefore,

$$(U(g_{(a,b)}) \circ \rho_A)(x,y) = U(g_{(a,b)})(\{x\}) = \{a\} = (\rho_A \circ O(g_{(a,b)}))(x,y) = \rho_A(a,b).$$

Thus, as before, $\rho_A = \tau_A$.

Case 3: $\rho_A(x,y) = \theta_A(x,y) = \{y\}$. Then $U(g_{(a,b)})(\rho_A(x,y)) = U(g_{(a,b)})(\{y\}) = \{b\}$, so

$$(U(g_{(a,b)}) \circ \rho_A)(x,y) = U(g_{(a,b)})(\{y\}) = \{b\} = (\rho_A \circ O(g_{(a,b)}))(x,y) = \rho_A(a,b).$$

Hence $\rho_A = \theta_A$.

In sum we have shown that $\rho_A \in \{\eta_A, \tau_A, \theta_A\}$. It remains to show that if $\rho_A = \eta_A$ (respectively τ_A, θ_A), then also $\rho_B = \eta_B$ (respectively τ_B, θ_B) for any set B.

If B is empty or a singleton then, as we have already observed, $\eta_B = \tau_B = \theta_B = \rho_B$. We may therefore take $w, z \in B$ with $w \neq z$. By our previous arguments $\rho_B \in \{\eta_B, \tau_B, \theta_B\}$. Define $h : A \to B$ by

$$h(c) = \begin{cases} w & \text{if } c = x, \\ z & \text{otherwise.} \end{cases}$$

Thus we have $O(h)(x,y) = (w,z)$.

In case $\rho_A(x,y) = \eta_A(x,y) = \{x,y\}$ then $U(h)(\rho_A(x,y)) = U(h)(\{x,y\}) = \{w,z\}$, and, again by naturality,

$$(U(h) \circ \rho_A)(x,y) = U(h)(\{x,y\}) = \{w,z\} = (\rho_B \circ O(h))(x,y) = \rho_B(w,z).$$

Thus $\rho_B(w,z) = \eta_B(w,z)$, and hence also $\rho_B = \eta_B$, by our previous work.

In case $\rho_A(x,y) = \tau_A(x,y) = \{x\}$ then $U(h)(\rho_A(x,y)) = U(h)(\{x\}) = \{w\}$, so

$$(U(h) \circ \rho_A)(x,y) = U(h)(\{x\}) = \{w\} = (\rho_B \circ O(h))(x,y) = \rho_B(w,z),$$

so $\rho_B = \tau_B$.

Finally, if $\rho_A(x,y) = \theta_A(x,y) = \{y\}$ then $U(h)(\rho_A(x,y)) = U(h)(\{y\}) = \{z\}$, so

$$(U(h) \circ \rho_A)(x,y) = U(h)(\{y\}) = \{z\} = (\rho_B \circ O(h))(x,y) = \rho_B(w,z).$$

Thus, whether ρ agrees with η, τ or θ on A it agrees correspondingly on B so that ρ is truly equal to one of our three transformations. $\square$