HOMOLOGICAL ALGEBRA HOMEWORK 1

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6) Let $\mathcal{C}$ be the subcategory of Sets containing all objects but only the identity morphisms. Let $F = G$ be the inclusion functor of $\mathcal{C}$ into Sets. $\mathrm{Nat}(F, G)$ is not a set.

PROOF: Any set S has a unique power set $\mathcal{P}(S)$ and no two distinct sets have the same power set. Hence, the map $S \mapsto \mathcal{P}(\mathcal{P}(S))$ is an injection of $\mathrm{obj}(\mathcal{C})$ into itself.

For any set S let $f : \mathcal{P}(\mathcal{P}(S)) \longrightarrow \mathcal{P}(\mathcal{P}(S))$ be a bijection other than $\mathrm{id}_{\mathcal{P}(\mathcal{P}(S))}$. We can always choose a bijection other than $\mathrm{id}_{\mathcal{P}(\mathcal{P}(S))}$ because $|\mathcal{P}(\mathcal{P}(S))| \geq 2$. For any set A define

$$\eta_A := \begin{cases} \mathrm{id}_A & \text{if } A \neq \mathcal{P}(\mathcal{P}(S)) \\ f & \text{if } A = \mathcal{P}(\mathcal{P}(S)). \end{cases}$$

Then for any set A, the following commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(\mathrm{id}_A)=\mathrm{id}_{F(A)}"] \arrow[d, "\eta_A"'] & F(A) \arrow[d, "\eta_A"] \\ G(A) \arrow[r, "G(\mathrm{id}_A)=\mathrm{id}_{G(A)}"'] & G(A) \end{tikzcd}$$

since $F(A) = G(A) = A$. We will always be mapping from $F(A) = A$ onto itself by $F(\mathrm{id}_A) = \mathrm{id}_{F(A)} = \mathrm{id}_A$ because the only morphisms in $\mathcal{C}$ are the identity morphisms. Hence, for any set A the preceding square is the most general we can consider.

As each η_A is invertible, $\eta = (\eta_A)_{A \in \mathrm{obj}(\mathcal{C})}$ is a natural isomorphism from F to G. Each set S gives a different natural isomorphism from F to G, so that we can embed a copy of $\mathrm{obj}(\mathcal{C})$ into $\mathrm{Nat}(F, G)$. Since $\mathrm{obj}(\mathcal{C})$ is not a set, $\mathrm{Nat}(F, G)$ is not a set.