# HOMOLOGICAL ALGEBRA: HOMEWORK 1

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**5)** Let $G$ and $H$ be groups considered as 1-object categories.

**(a)** Show that functors between $G$ and $H$ correspond to group homomorphisms.

**(b)** For which pairs of functors, $E, F : G \to H$ do there exist natural transformations $\eta : E \to F$?

Solution

**(a)** *Proof.* For simplicity, let $G$ denote the single object in the category $G$ (likewise for $H$). Let $E : G \to H$ be a functor. Then for $g, h \in \mathrm{Hom}_G(G, G)$ we have $E(g) \in \mathrm{Hom}_H(E(G), E(G)) = \mathrm{Hom}_H(H, H)$. Furthermore, $E(g \circ h) = E(g) \circ E(h)$. Hence, $E$ can be realized as the group homomorphism $\tilde{E} : G \to H : g \mapsto E(g)$. Conversely, if $\tilde{F} : G \to H$ is a group homomorphism, define a mapping of categories $F : G \to H : G \mapsto H; g \mapsto \tilde{F}(g)$. Since

$$F(g \circ h) = \tilde{F}(g \circ h) = \tilde{F}(g) \circ \tilde{F}(h) = F(g) \circ F(h),$$

$F$ is a functor. Hence functors between $G$ and $H$ correspond to group homomorphisms. $\square$

**(b)** Suppose that for functors $E, F : G \to H$ there is natural transformation $\eta : E \to F$. Then for every $g \in \mathrm{Hom}_G(G, G)$ the diagram

$$\begin{tikzcd} E(G) \arrow[r, "E(g)"] \arrow[d, "\eta_G"'] & E(G) \arrow[d, "\eta_G"] \\ F(G) \arrow[r, "F(g)"'] & F(G) \end{tikzcd}$$

commutes. That is, $\eta_G E(g) = F(g)\eta_G$ for all $g$. Since $\eta_G \in \mathrm{Hom}_H(E(G), F(G)) = \mathrm{Hom}_H(H, H)$, it corresponds to a group element and is invertible. Thus $E(g) = \eta_G^{-1} F(g) \eta_G$ for all $g$. Translating this statement into the language of groups gives that $\tilde{E}(g) = \eta_G^{-1} \tilde{F}(g) \eta_G$, hence these homomorphisms are conjugate. If $\tilde{F}$ and $\tilde{E}$ are conjugate group homomorphisms then $\tilde{E}(g) = \eta^{-1} \tilde{F}(g) \eta$ for some $\eta \in H$ Letting $\eta = \eta_G$ then defines a natural transformation between the functors $E$ and $F$.

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