Math 8174, Homework 1

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#5. Let Div be the full subcategory of Ab consisting of divisible groups. Show the natural map $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic but not 1-1.

Proof. Let G be a divisible group, and $f, g \in \operatorname{Hom}(G, \mathbb{Q})$ be such that $\pi \circ f = \pi \circ g$. Since f and g are homomorphisms targeted in an abelian group, we may add them. Our condition on f and g implies $\pi \circ (f - g) = 0$. Define $h(x) = f(x) - g(x)$. For the composition to be zero, it must be that $\operatorname{im}(h) \subset \ker(\pi) = \mathbb{Z}$. Now let $x \in G$ satisfy $h(x) = a \neq 0$. Write $\frac{x}{2h(x)}$ for the element of G which when multiplied by $2h(x)$ yields x (remember $h(x) \in \mathbb{Z}$). We have

$$h\left(\frac{x}{2h(x)}\right) = h\left(\frac{x}{2a}\right) = \frac{a}{2a} = \frac{1}{2}.$$

But $\frac{1}{2} \notin \mathbb{Z}$. Therefore $\pi \circ h \neq 0$ which is a contradiction.

That π is not 1-1 follows from the existence of positive integers.

□