HOMOLOGICAL ALGEBRA HOMEWORK I

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Weibel A.1.4 Show that
$$\mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i).$$

Solution

We will show the stronger result that the above is indeed a natural isomorphism of the functors
$$H_1 \colon \mathcal{C}^{\mathbf{op}} \times \mathcal{C}^I \to \mathbf{Sets}$$
and
$$H_2 \colon \mathcal{C}^{\mathbf{op}} \times \mathcal{C}^I \to \mathbf{Sets}$$
where $\mathcal{C}^{\mathbf{op}} \times \mathcal{C}^I$ is a category with objects $(A, \overline{C})$ with $\overline{C} = (C_i)_{i \in I}$ and where A, C_i are objects in $\mathcal{C}$ for all i. The morphisms of $\mathcal{C}^{\mathbf{op}} \times \mathcal{C}^I$ are pairs
$$(f, \overline{g}) \colon (A, \overline{C}) \to (B, \overline{D})$$
where $f \colon B \to A$ and $\overline{g} = (g_i)_{i \in I}$ with $g_i \colon C_i \to D_i$ and where f, g_i are morphisms in $\mathcal{C}$. The functor H_1 is defined by
$$(A, \overline{C}) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i)$$
$$\Big[(f, \overline{g}) \colon (A, \overline{C}) \to (B, \overline{D}) \Big] \mapsto \left[\begin{array}{c} H_1(f, \overline{g}) \colon \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i) \to \mathrm{Hom}_{\mathcal{C}}(B, \prod_{i \in I} D_i) \\ x \mapsto g_i^* \circ x \circ f \end{array} \right]$$
where $x \in \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i)$. So that this functor is well-defined, we choose a fixed representative for each isomorphism class $\prod_{i \in I} C_i$. In the above definition it remains to define the map
$$g_i^* \colon \prod_{i \in I} C_i \to \prod_{i \in I} D_i.$$
Let the morphisms
$$\pi_i^{\overline{C}} \colon \prod_{i \in I} C_i \to C_i$$
and
$$\pi_i^{\overline{D}} \colon \prod_{i \in I} D_i \to D_i$$
be the projection maps that are implicit in the definition of the products $\prod_{i \in I} C_i$ and $\prod_{i \in I} D_i$, respectively. We have that
$$g_i \circ \pi_i^{\overline{C}} \colon \prod_{i \in I} C_i \to D_i$$
and by the universal property of products there exists a map to the product of the D_i, which is the map we denote by
$$g_i^* \colon \prod_{i \in I} C_i \to \prod_{i \in I} D_i,$$
and g_i^* is unique such that
$$\pi_i^{\overline{D}} \circ g_i^* = g_i \circ \pi_i^{\overline{C}}.$$
This completes the description of H_1.

The functor H_2 is defined by

$$(A, \overline{C}) \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i)$$

$$\Big[(f, \overline{g}) \colon (A, \overline{C}) \to (B, \overline{D})\Big] \mapsto \left[\begin{array}{c} H_1(f, \overline{g}) \colon \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i) \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(B, D_i) \\ (y_i)_{i \in I} \mapsto g_i \circ y_i \circ f \end{array} \right]$$

So that this functor is well-defined, we will take the product to be Cartesian product. We now define a collection of of isomorphisms (in the category **Sets** these are bijections):

$$\eta_{(A, \overline{C})} \colon \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i) \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i)$$

indexed by the objects $(A, \overline{C})$ in the category $\mathcal{C}^{\mathbf{op}} \times \mathcal{C}^I$. If the domain and codomain are clear from the context we will just denote these maps by η. The map $\eta_{(A, \overline{C})}$ is defined by

$$x \mapsto (\pi_i^{\overline{C}} \circ x)_{i \in I}$$

To show $\eta_{(A, \overline{C})}$ is a bijection, we consider the map

$$\eta^{-1}_{(A, \overline{C})} \colon \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i) \to \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i)$$

defined by

$$(y_i)_{i \in I} \mapsto y^*$$

where y^* is the map to the product induced by the maps y_i, which is unique such that $\pi_i^{\overline{C}} \circ y^* = y_i$. Then we have

$$\eta \circ \eta^{-1}((y_i)_{i \in I}) = \eta(y^*) = (\pi_i^{\overline{C}} \circ y^*)_{i \in I} = (y_i)_{i \in I}$$

and also

$$\eta^{-1} \circ \eta(x) = \eta^{-1}((\pi_i^{\overline{C}} \circ x)_{i \in I}) = x$$

where the last equality holds by the uniqueness of the map x, which clearly satisfies the condition of the universal property. It follows that $\eta_{(A, \overline{C})}$ is a bijection. To verify that $\eta \colon H_1 \to H_2$ is a natural isomorphism, it remains to check the naturality condition, i.e. that

$$\eta_{(B, \overline{D})} \circ H_1(f, \overline{g}) = H_2(f, \overline{g}) \circ \eta_{(A, \overline{C})}.$$

Let $x \in \mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i)$. Then we compute

$$\begin{aligned} (\eta_{(B, \overline{D})} \circ H_1(f, \overline{g}))(x) &= \eta_{(B, \overline{D})}(g_i^* \circ x \circ f) \\ &= (\pi_i^D \circ (g_i^* \circ x \circ f))_{i \in I} \\ &= ((\pi_i^D \circ g_i^*) \circ x \circ f)_{i \in I} && \text{by associativity of composition} \\ &= ((g_i \circ \pi_i^{\overline{C}}) \circ x \circ f)_{i \in I} && \text{by the definition } g_i^* \\ &= (g_i \circ \pi_i^{\overline{C}} \circ x \circ f)_{i \in I}. \end{aligned}$$

We also compute

$$\begin{aligned} (H_2(f, \overline{g}) \circ \eta_{(A, \overline{C})})(x) &= H_2((\pi_i^{\overline{C}} \circ x)_{i \in I}) \\ &= (g_i \circ (\pi_i^{\overline{C}} \circ x) \circ f)_{i \in I} \\ &= (g_i \circ \pi_i^{\overline{C}} \circ x \circ f)_{i \in I} && \text{by associativity.} \end{aligned}$$

Thus we have that η is a natural isomorphism between H_1 and H_2, which was to be demonstrated. $\square$