Problem 2. (Hower, Jones). *Show that the category C of finite abelian groups is equivalent to its opposite category C^{op}, but that this fails for the category, T of torsion abelian groups.*

Solution: It suffices to find a pair of contravariant functors $F, G : C \to C$ such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functor on C.

Given an abelian group A, define A^* to be the set of group homomorphisms from A into the multiplicative group of the unit circle, S^1. A^* has the structure of an abelian group under pointwise multiplication. I claim that A^* is a finite group; moreover, that $|A^*| = |A|$.

To see this, write A as a direct product of cyclic groups, $A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. For each i, let g_i generate $\mathbb{Z}_{n_i}$. Now any homomorphism $\phi : A \to S^1$ is determined by the images of these generators, $\phi(g_1), \ldots, \phi(g_k)$.

It is an elementary group theory fact that the order of $\phi(g_i)$ must divide the order of g_i. In S^1, there are at most n_i such elements, namely the roots of unity. Therefore, there are at most $\prod n_i = |A|$ choices available, and thus at most $|A|$ homomorphisms. But in fact, each of these choices gives rise to a homomorphism because the only relations in A are of the form $g_i^{n_i} = 1$. This proves the claim.

We now have the makings of the object part of a functor from C to C. The construction of the morphism part follows.

Suppose that A and B are abelian groups, and $f : A \to B$ is a group homomorphism. Define a map $f^* : B^* \to A^*$ by setting $f^*(\psi) = \psi \circ f$. To prove that this association is functorial, there are a few things one must check:

(i) f^* is a homomorphism:

This is due to the fact that composition of homomorphisms distributes over addition of homomorphisms.

(ii) $(Id_A)^* = Id_{A^*}$:

Composing a function with the identity morphism does not change it.

(iii) $(f \circ g)^* = g^* \circ f^*$:

$(f \circ g)^*(\psi) = \psi \circ (f \circ g) = (\psi \circ f) \circ g = (f^*(\psi)) \circ g = g^*(f^*(\psi)) = [g^* \circ f^*](\psi)$.

Thus * is a contravariant functor from C to C. I claim that it is a self-inverse equivalence of categories. To prove this, I must construct a natural isomorphism $\eta_A : A \to A^{**}$.

To this end, declare $a^{**} \in A^{**}$ to be the homomorphism that acts on elements $\phi \in A^*$ in the following way:

$$a^{**}(\phi) = \phi(a).$$

(a^{**} is "evaluation at a"). Define $\eta_A(a) = a^{**}$. η_A is a homomorphism, because elements of A^* are homomorphisms.

To see that η_A is injective, think of A as presented previously. Let $a = g_1^{m_1} \cdots g_k^{m_k} \in A$ and suppose that $\eta_A(a) = 1$. This means that for every homomorphism $\phi \in A^*$, $\phi(a) = 1$. This is true, in particular, for the homomorphisms ϕ_i which send g_i to a primitive n_i^{th} root of unity and send all other generators to 1. The only way that every single one of these could map a to 1 is if a was already the identity element. Thus η_A is injective (hence an isomorphism, due to the claim.)

It remains to be shown that η_A is natural.

Let $f : A \to B$ be a homomorphism of finite abelian groups. I must show that

$$f^{**} \circ \eta_A = \eta_B \circ f$$

Both sides of this equation are functions from A to B^{**}, and so they will be equal if

$$(f^{**} \circ \eta_A)(a) = (\eta_B \circ f)(a)$$

for an arbitrary $a \in A$. Both sides of *this* equation are functions from B^* to S^1, and so they will be equal if

$$[(f^{**} \circ \eta_A)(a)](\psi) = [(\eta_B \circ f)(a)](\psi)$$

for an arbitrary $\psi \in B^*$. Let's unravel the left side:

$$\begin{aligned}
[(f^{**} \circ \eta_A)(a)](\psi) &= [f^{**}(\eta_A(a))](\psi) \\
&= [f^{**}(a^{**})](\psi) \\
&= [a^{**} \circ f^*](\psi) \\
&= a^{**}(f^*(\psi)) \\
&= a^{**}(\psi \circ f) \\
&= (\psi \circ f)(a) \\
&= \psi(f(a)) \\
&= f(a)^{**}(\psi) \\
&= [\eta_B(f(a))](\psi) \\
&= [(\eta_B \circ f)(a)](\psi)
\end{aligned}$$

This proves that μ_A is natural, which establishes the equivalence between C and C^{op}.

Now let T be the category of all abelian Torsion groups. We want to show that T is not equivalent to T^{op}. For a given group, we want to build its subgroup lattice categorically. Let A be an abelian torsion group. Consider all monomorphisms of groups into A. Let $\beta : B \mapsto A$ and $\gamma : C \mapsto A$ be monomorphisms. Define an equivalence relation $\sim$ as follows, $\beta \sim \gamma$ iff $\exists \alpha : B \mapsto C(\beta = \gamma\alpha \wedge \gamma = \beta\alpha^{-1})$. This means that two monomorphisms are equivalent iff they have the same image. Now consider the ordering, $[\beta] \leq [\gamma]$ iff $\exists \alpha(\beta = \gamma\alpha)$. This order defined on equivalence classes corresponds to the inclusion order on images. This ordering gives rise to the subgroup lattice of any group. Similarly, you could build the subgroup lattice by considering the epimorphisms out of the group A. Here, two epimorphisms are equivalent iff they have the same kernel.

Now assume that T is equivalent to T^{op}. This means that there exists two contravariant functors, F and G, from $T \mapsto T$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to Id_T. Since the subgroup lattice is determined categorically and a contravariant functor would switch kernels and images of homomorphisms, the functor F should take the subgroup lattice and any torsion abelian group and flip it. However, consider the torsion abelian group $\mathbb{Z}_{p^\infty}$. Its subgroup lattice is an infinite ascending chain together with a point above the chain. The functor F would then flip this subgroup lattice and it would become an infinite descending chain together with a zero point.

This is impossible with any torsion abelian group since and element in the group would generate a subgroup of finite order which would result in a subgroup directly above zero. This does not happen. Therefore, the functors F and G do not exist and T is not equivalent to T^{op}.

$\square$