Exercise 1.1.6: Let Γ be a finite graph with V vertices $(v_1, v_2, \ldots, v_V)$ and E edges $(e_1, e_2, \ldots, e_E)$. If we orient the edges, we can form the incidence matrix of the graph. This is a $V \times E$ matrix whose (ij)-entry is $+1$ if the edge e_j starts at v_i, -1 if e_j ends at v_i, and 0 otherwise. Let C_0 be the free R-module on the vertices, C_1 the free R-module on the edges, $C_n = 0$ if $n \neq 0, 1$, and $d : C_1 \to C_0$ the incidence matrix. If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimension 1 and $E - V + 1$ respectively.

Proof: Notice first $H_1(\Gamma) = \ker(d)$ and $H_0(\Gamma) = C_0/\mathrm{im}(d)$. Now for any $a_1v_1 + a_2v_2 + \cdots + a_vv_V \in C_0$ can be rewritten: $a_1(v_1 - v_V) + a_2(v_2 - v_V) \cdots + a_{V-1}(v_{V-1} - v_V) + (a_1 + a_2 + \cdots + a_V)v_V$. And if we have that $b_1(v_1 - v_V) + \cdots + b_{V-1}(v_{V-1} - v_V) + b_Vv_V = 0$ then, for $i = 1, 2, \ldots, (V - 1)$, the vertex v_i only appears once so $b_i = 0$ but this implies $b_V = 0$. Hence the set $\{v_i - v_V, v_V\}_{i=1}^{V-1}$ is a free basis of C_0. Therefore we have that:

$$C_0 = \left(\bigoplus_{i=1}^{V-1} R(v_i - v_V)\right) \oplus Rv_V.$$

Since Γ is connected we may find a tree $T \subseteq \Gamma$, that contains all of the vertices. For each $i = 1, 2, \ldots, V$ there is a path in T from v_V to v_i passing through edges $e_{i1}, e_{i2}, \ldots, e_{ik}$. If when we traverse this path from v_V to v_i we travel along the edge e_{ij} from tail to head we let $\eta_{ij} = 1$, and if we traverse e_{ij} from head to tail we let $\eta_{ij} = -1$. Define $c_i = \eta_{i1}e_{i1} + \cdots + \eta_{ik}e_{ik}$. Note that $dc_i = v_i - v_V$. Thus $v_i - v_V \in \mathrm{im}(d)$ for $i = 1, 2, \ldots, (V - 1)$. This mean that:

$$\left(\bigoplus_{i=1}^{V-1} R(v_i - v_V)\right) \subseteq \mathrm{im}(d).$$

Since C_0 is free on the set $\{v_i\}_{i=1}^{V}$ we may define a map $\epsilon : C_0 \to R : v_i \mapsto 1$ for all $i = 1, 2, \ldots, V$. Since R is a free ring on the generator 1 we may define $s : R \to C_0$ by $s(1) = v_V$. Notice that for any $r \in R$ we have $(\epsilon \circ s)(r) = r(\epsilon \circ s)(1) = r$. Thus $\epsilon \circ s = 1_R$, so ϵ maps Rv_V isomorphically onto R. For each edge e_j with $j = 1, 2, \ldots, E$ we have $(\epsilon \circ d)e_j = 0$ hence

$$\left(\bigoplus_{i=1}^{V-1} R(v_i - v_V)\right) \subseteq \mathrm{im}(d) \subseteq \ker(\epsilon) \subseteq \left(\bigoplus_{i=1}^{V-1} R(v_i - v_V)\right).$$

Therefore we have that

$$\mathrm{im}(d) = \left(\bigoplus_{i=1}^{V-1} R(v_i - v_V)\right),$$

and hence

$$H_0(\Gamma) = C_0/\mathrm{im}(d) \cong Rv_V$$

is a free R-module on one generator.

Recall that a tree with V vertices has $V - 1$ edges, so there are $E - V + 1$ edges not in T. For each edge, e_j, going from v_{ji} to v_{jk} that is not in the tree T we define the chain $z_j = c_{ji} + e_j - c_{jk} \in C_1$. If we have $d_1z_1 + \cdots + d_{(E-V+1)}z_{(E-V+1)} = 0$, then each edge e_i that is in z_i but not T shows up only once in this sum so its coefficient, d_i, must be 0. Thus all of the d_i's are 0, and the z_i's are independent. Note that $dz_j = dc_{ji} + de_j - dc_{jk} = v_{ij} - v_V + v_{jk} - v_{ij} + v_V - v_{jk} = 0$. Hence we have

$$\bigoplus_{i=1}^{E-V+1} Rz_i \subseteq \ker(d).$$

If any edge e_i going from v_{ij} to v_{ik} is in T then $e_i = c_{ik} - c_{ij}$, and if it is not in T then $e_i = c_{ik} + z_i - c_{ij}$. Hence $\{c_i\}_{i=1}^{V-1} \cup \{z_j\}_{j=1}^{E-V+1}$ spans C_1. So we have,

$$C_1 = \left(\bigoplus_{i=1}^{E-V+1} Rz_i\right) + \mathrm{Span}_R\left(\{c_j\}_{j=1}^{V-1}\right).$$

Define $A = \mathrm{Span}_R\left(\{c_j\}_{j=1}^{V-1}\right)$, now define a map $\rho : \mathrm{im}(d) \to A : v_i - v_V \mapsto c_i$. Note that $(\rho \circ d|_A) = 1_A$. Thus $d|_A$ is injective. If $x \in C_1$ we can write $x = z + c$ with $c \in A$ and $z \in \bigoplus_{i=1}^{E-V+1} Rz_i$. Now if $x \in \ker(d)$ then $0 = dx = dz + dc = dc$, but since $d|_A$ is injective, and $c \in A$, we must have $c = 0$. Hence $x = z \in \bigoplus_{i=1}^{E-V+1} Rz_i$. This means that $\ker(d) \subseteq \bigoplus_{i=1}^{E-V+1} Rz_i$. Therefore we have shown that $\ker(d) = \bigoplus_{i=1}^{E-V+1} Rz_i$, thus giving the desired results:

$$H_1(\Gamma) = \ker(d) = \bigoplus_{i=1}^{E-V+1} Rz_i$$

is a free R-module of rank $E - V + 1$.$\square$