SET THEORY
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Russell's Paradox shows that there is no set of all sets. Another argument for this is Cantor's Paradox: if $\mathcal{S}$ is the set of all sets, then $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$.

(a) Why is $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$?

Any element of $\mathcal{P}(S)$ is a set, so would have to be an element of $\mathcal{S}$. Hence $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$.

(b) Why should Cantor think this is a paradox?

If $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$, then $|\mathcal{P}(\mathcal{S})| \leq |\mathcal{S}|$. Together with Cantor's Theorem, $|\mathcal{S}| < |\mathcal{P}(\mathcal{S})|$, we conclude $|\mathcal{S}| < |\mathcal{S}|$, which is false.

2. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ implies $A \subseteq B$.

If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then since $A \in \mathcal{P}(A)$ we get $A \in \mathcal{P}(B)$, hence $A \subseteq B$.

3.

(a) What is the recursive definition of exponentiation of natural numbers?

$$\begin{array}{lrl} \text{(IV)} & m^0 & := 1 \\ \text{(RR)} & m^{S(n)} & := m^n \cdot m \end{array}$$

(b) Prove that $(m \cdot n)^k = m^k \cdot n^k$ for all $m, n, k \in \mathbb{N}$. (You may use any valid arithmetic results that concern *addition* and *multiplication*, but identify which results you are using.)

We prove this by induction on k.

(Basis of induction, $k = 0$)

$$\begin{aligned} (m \cdot n)^0 &= 1 && ((\text{IV}), \text{EXP}) \\ &= 1 \cdot 1 && (1 \text{ is a multiplicative unit}) \\ &= m^0 \cdot n^0 && ((\text{IV}), \text{EXP}). \end{aligned}$$

(Inductive step) Assume $(m \cdot n)^k = m^k \cdot n^k$.

$$\begin{aligned} (m \cdot n)^{S(k)} &= (m \cdot n)^k \cdot (m \cdot n) && ((\text{RR}), \text{EXP}) \\ &= (m^k \cdot n^k) \cdot (m \cdot n) && (\text{Inductive Hypothesis}) \\ &= (m^k \cdot m) \cdot (n^k \cdot n) && (\text{Assoc. \& Comm. Laws for } \cdot) \\ &= m^{S(k)} \cdot n^{S(k)} && ((\text{RR}), \text{EXP}). \end{aligned}$$

4. A sequence $\langle a_0, a_1, a_2, \ldots \rangle$ is ***strictly increasing*** if $a_0 < a_1 < a_2 < \cdots$. Show that the set of strictly increasing sequences of natural numbers is uncountable.

I meant for these sequences to be infinite, but our book allows "sequence" to refer to either finite or infinite sequences. Here are solutions for both interpretations of the problem.

Solution 1. (Sequences may be finite or infinite.) Consider the function $F\colon \mathrm{Seq} \to \mathcal{P}(\mathbb{N})\colon \sigma \mapsto \mathrm{ran}(\sigma)$ from the set of strictly increasing sequences to $\mathcal{P}(\mathbb{N})$, which maps a sequence $\sigma = \langle a_0, a_1, \ldots \rangle$ to its range $\{a_0, a_1, \ldots\}$. This is a bijection, whose inverse is the function which converts a subset $S \subseteq \mathbb{N}$ into a sequence by ordering its elements. (F^{-1} can be defined by recursion.)

Solution 2. (Sequences must be infinite.) You can try to do the same thing here, but you only obtain a bijection $F\colon \mathrm{Seq}_\infty \to \mathcal{P}_\infty(\mathbb{N})$ from the set of infinite strictly increasing sequences to the set of infinite subsets of $\mathbb{N}$. Thus we need to show that $\mathbb{N}$ has uncountably many infinite subsets.

Suppose instead that $\mathcal{P}_\infty(\mathbb{N})$ is countable. Then $\mathcal{P}_{\text{fin}}(\mathbb{N})$, the set of finite subsets of $\mathbb{N}$, must also be countable, since the function $G\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathcal{P}_\infty(\mathbb{N})$ that maps a finite set to its complement is an injection. But if both $\mathcal{P}_{\text{fin}}(\mathbb{N})$ and $\mathcal{P}_\infty(\mathbb{N})$ are countable, then so is their union, $\mathcal{P}(\mathbb{N})$, which we know is not true. Thus $\mathcal{P}_\infty(\mathbb{N})$ is uncountable.