

## SET THEORY (MATH 4730)

### SUMMARY OF TOPICS FROM 3/9/09-4/29/09

#### VI. Ordinals.

- (a) Well-ordered sets. (Definition. Uniqueness properties from Section 6.1.3.)
- (b) Ordinal numbers. (Definition. Examples. Successor vs. limit ordinals. Ordinals are well-ordered. Order type. Natural numbers are exactly the finite ordinals.  $\alpha \notin \alpha$  provable without Foundation.)
- (c) Axiom of Replacement.
- (d) Recursion Theorem.
- (e) Transfinite induction.
- (f) Ordinal arithmetic. (Definitions, properties, order-theoretic interpretations.)
- (f) Normal functions. (= continuous +  $<$ -preserving.)
- (g) Cantor normal form.
- (h)  $\varepsilon_0$  (the least fixed point  $\omega^x$ ).

#### VII. Cardinals.

- (a) Definition (= initial ordinal) and notation. ( $\aleph$  is a normal function.)
- (b) Hartogs number of a set.
- (c) Definitions for cardinal arithmetic.

#### VIII. Axiom of Choice.

- (a) ZF proves that  $A$  can be well-ordered iff  $\mathcal{P}(A)$  has a choice function.
- (b) Within ZF, AC is equivalent to:
  - (i) Well-Ordering Principle.
  - (ii) Zorn's Lemma.
  - (iii) Every surjective function has a right inverse.
  - (iv) Every partition has an SDR.
  - (v) Every set is equipotent with a (unique) cardinal.
  - (vi) Any two sets have comparable cardinalities.
- (c) AC implies that
  - (i) every infinite set has a countably infinite subset.
  - (ii) every Dedekind finite set is finite.
  - (iii) a countable union of countable sets is countable.
  - (iv)  $2^{\aleph_0} \geq \aleph_1$ .
  - (v) every vector space has a basis.

#### IX. Cardinal Arithmetic.

- (a) Infinite sums and products.
- (b) Cofinality.
- (c) König's Theorem (+ the corollary  $\kappa^{\text{cf}(\kappa)} > \kappa$ ).
- (d) Regular and singular cardinals.

- (e) Successor cardinals are regular.  $\aleph_\omega$  is singular.
- (f) Within ZFC a cardinal  $\kappa$  may equal  $|\mathbb{R}| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$  iff  $\text{cf}(\kappa)$  is uncountable.
- (g) Cardinal exponentiation.
- (h) CH and GCH.

### General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

### Sample Problems.

- (1) (a) Define “ordinal”.  
 (b) Show that 0 is an ordinal and that the successor of an ordinal is an ordinal.  
 (c) Show that an element of an ordinal is an ordinal.
- (2) Explain why the collection of all ordinals is not a set.
- (3) Is the set of proper subsets of  $\omega$ , ordered by inclusion, an inductively ordered set? What conclusions, if any, can be drawn?
- (4) Show that the Hartogs number of a set is a cardinal.
- (5) Use transfinite induction to show that every vector space has a basis. (Hint: enumerate the vectors in  $V$  with an ordinal. Define a maximal independent subset of  $V$  by recursion. Prove that the subset is a basis by induction.)
- (6) Show that if  $\kappa$  is infinite, then  $2^\kappa = \kappa^\kappa$ .