Quantifiers.

In these notes we will discuss how to determine the truth of formulas of the form

$$(\dagger) \qquad \varphi = \forall x_1 \ \cdots \ \exists x_n \ F(x_1, \ldots, x_n),$$

which consist of a string $\forall x_1 \ \cdots \ \exists x_n$ of quantified variables followed by a quantifier-free formula $F(x_1, \ldots, x_n)$ whose variables are among those that have been quantified. So, we will discuss formulas like

$$\exists x \ (x = x^2) \quad \text{or} \quad \forall x \ \exists y \ (x < y) \quad \text{or} \quad \forall x \ \forall y \ \exists z \ ((x^2 + y^2 = z^2 + 1) \lor (x^2 + y^2 = z^2)),$$

but postpone the discussion of formulas like

$$\underbrace{\exists x \ (x + y = 0)}_{\text{unquantified variable } y} \qquad \text{or} \qquad \underbrace{\forall x \ ((0 < x) \to \exists y \ ((0 < y) \land (y < x))).}_{\text{one quantifier is not at the front}}$$

When applied to ordered sets, the formula $\exists t \ \forall x \ (x \leq t)$ expresses the fact that there is a top element. Some ordered sets have a top element and others do not, so the truth of this formula depends on which structure is referred to. Let's define "structure":

Definition 1. A *structure* for a language L is a set equipped with a fixed interpretation of each of the nonlogical symbols of L.

Here are some examples of structures for different languages:
$V = \langle \{\text{a universe of sets}\}; \in \rangle$.
$\mathbb{N} = \langle \{\text{natural numbers}\}; +, -, 0, \cdot, 1, S \rangle$.
$\mathbb{R} = \langle \{\text{real numbers}\}; +, -, 0, \cdot, 1, < \rangle$.
$\mathbb{Z}_n = \langle \{\text{integers mod } n\}; +, -, 0, \cdot, 1 \rangle$.

Quantifier games. Given a formula φ of type $(\dagger)$ and a structure $\mathbb{A} = \langle A; +, -, <, \ldots \rangle$ in the same language as φ there is a game between the quantifiers $\exists$ and $\forall$ which decides the truth of φ in $\mathbb{A}$. In this game, $\exists$ (also known as Eloise, or The Prover) tries to prove that φ is true in $\mathbb{A}$ while $\forall$ (also known as Abélard, or The Refuter) tries to prove that φ is false in $\mathbb{A}$.

The sequence of play is determined by the sequence of quantifiers in the formula. For example, if $\varphi = \forall x \ \exists y \ \exists z \ ((x < y) \land (z < x))$, then one play of the game involves one turn by $\forall$ followed by two turns by $\exists$. A player takes a turn by selecting an element of the structure to substitute for the quantified variable. For example if $\mathbb{A} = \langle \mathbb{Z}; < \rangle$, then $\forall$ may select $1 \in \mathbb{Z}$ to substitute for x, then $\exists$ may select $0 \in \mathbb{Z}$ to substitute for y, and finally $\exists$ may select $7 \in \mathbb{Z}$ to substitute for z. Now we delete the quantifiers from φ and make the substitutions: $((1 < 0) \land (7 < 1))$. This is false in the structure $\langle \mathbb{Z}; < \rangle$, so it counts as a win for The Refuter, $\forall$. (Note: each time a quantifier chooses an value to substitute for its variable, it is allowed to know all choices made previously by either player.)

The formula φ is false in $\mathbb{A}$ if $\forall$ has a *winning strategy*, which means a strategy to force a win every time the game is played. The formula is true in $\mathbb{A}$ if $\exists$ has a winning strategy.

Although $\forall$ won the instance of the game described in the previous paragraph, $\forall$ does not have a winning strategy; rather, $\exists$ does. One strategy for $\exists$ is: "whatever $\forall$ chooses for the value of x, choose $y = x + 1$ and $z = x - 1$".

Zermelo's Theorem (from Game Theory) guarantees that one of $\exists$ or $\forall$ must have a winning strategy. It may not be easy to find a winning strategy. Indeed, a winning strategy for $\exists$ is a proof that φ is true, while a winning strategy for $\forall$ is a proof that φ is false, so finding winning strategies is the same thing as finding proofs for formal statements. This means that discovering winning strategies requires creativity and effort.

Exercises. Decide the truth or falsity of the given formula in the given structure. In each example, describe a winning strategy.

(1) $\forall x\ (x < x^2)$, in $\mathbb{R}$.

False. A winning strategy for $\forall$ is to assign $1/2$ to x.

(2) $\exists x\ (\neg(x^{31} = x))$, in $\mathbb{Z}_{31}$.

False. A winning strategy for $\forall$ is to do nothing. $\exists$ cannot choose a value for x to make the statement true, because this would be an x for which $x^{31} \not\equiv x \pmod{31}$, contradicting Fermat's Little Theorem.

(3) $\exists x\ \forall y\ (\neg(x = S(y)))$, in $\mathbb{N}$.

True. A winning strategy for $\exists$ is to assign 0 to x.

(4) $\forall x\ \exists y\ \forall z\ ((x \in y) \wedge ((z \in y) \rightarrow (z = x)))$, in V.

True. If $\forall$ assigns the value a to x, then $\exists$ can win by assigning $\{a\}$ to y.

(5) $\forall w\ \exists x\ \forall y\ \exists z\ (w^2 + x^2 = y^2 + z^2)$, in $\mathbb{R}$.

False. A winning strategy for $\forall$ is to assign a value to w randomly, let $\exists$ assign a value to x, then assign a value for y so that $y^2 > w^2 + x^2$. Then there is no value $\exists$ can assign to z which will make the formula true.

(6) $\forall v\ \exists w\ \exists x\ \exists y\ \exists z\ (v = w^2 + x^2 + y^2 + z^2)$, in $\mathbb{N}$.

True. Lagrange's Theorem states that every natural number is a sum of four squares of natural numbers. The strategy for $\exists$ is to look at which value $\forall$ selects for v and then look through all smaller numbers until she finds w, x, y, z that work. (Lagrange's Theorem guarantees that they exist.) Try this for $v = 23$.

(7) $\forall u\ \exists v\ \forall w\ \exists x\ \forall y\ \exists z\ ((u - v)(w - x)(y - z) = 1)$, in $\mathbb{Z}_{10}$.

True. A winning strategy for $\exists$ is to assign $v = u - 1, x = w - 1$ and $z = y - 1$.