Prenex form.

A formula φ is in *prenex form* if all the quantifiers are at the front. We learned in class how to put a formula in prenex form. The procedure was based on the rules:

- $\neg(\exists x\ \alpha) \equiv \forall x\ (\neg\alpha)$
- $\neg(\forall x\ \alpha) \equiv \exists x\ (\neg\alpha)$
- $((\exists x\ \alpha(x)) \wedge \beta) \equiv \exists x\ (\alpha(x) \wedge \beta)$ if x is not free in β
- $((\forall x\ \alpha(x)) \wedge \beta) \equiv \forall x\ (\alpha(x) \wedge \beta)$ if x is not free in β
- $((\exists x\ \alpha(x)) \vee \beta) \equiv \exists x\ (\alpha(x) \vee \beta)$ if x is not free in β
- $((\forall x\ \alpha(x)) \vee \beta) \equiv \forall x\ (\alpha(x) \vee \beta)$ if x is not free in β

The quantifier-free part of a formula in prenex form is called the *matrix* of the formula. It is always possible to put the matrix in disjunctive normal form, so every formula is equivalent to one of the form "(quantifiers) $\bigvee \bigwedge$ ($\pm$ atomic)".

In the "Quantifiers" handout we learned how to determine whether a sentence in prenex form is true in a structure. Today we will practice putting a formula in prenex form and testing the truth of a sentence in prenex form.

Exercises on Prenex Form. Put the following formulas in prenex form.

(1) $\exists y\ ((\exists x\ (0 < x)) \rightarrow (y < y^2))$

$$\exists y\ \forall x\ ((0 < x)) \rightarrow (y < y^2))$$

(2) $((\exists x\ (0 < x)) \wedge (\forall x\ (x \leq 1)))$

$$\exists x\ \forall y\ ((0 < x) \wedge (y \leq 1))$$

(3) $((\forall x\ P(x) \rightarrow \forall y\ Q(y)) \rightarrow (\forall x\ R(x) \rightarrow \forall y\ S(y)))$

$$\forall x\ \exists y\ \exists z\ \forall w\ ((P(x) \rightarrow Q(y)) \rightarrow (R(z) \rightarrow S(w)))$$

(4) You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time. (Hint #1: Use predicates $\text{Person}(x), \text{Time}(t)$, and $\text{Fool}(x, y, t)$ expressing "x is a person", "t is a time", and "x fools y at time t". Hint #2: Write the sentence is the most obvious way first, then change it to prenex form. Hint #3: Interpret "You can fool ..." to mean "Anyone can fool ...".)

$$\forall x\ \exists y\ \forall t\ \forall y'\ \exists t'\ \exists y''\ \exists t''\ (F(x, y, t) \wedge F(x, y', t') \wedge \neg F(x, y'', t''))$$

Exercises on Truth. Decide the truth or falsity of the given formula in $\mathbb{R}$. In each example, describe a winning strategy.

In the first two problems use the sentence[1]

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ ((0 < |x-1| < \delta) \rightarrow (|f(x) - f(1)| < \epsilon))$$

This is the statement that the function f is continuous at $x = 1$. In these problems the structure $\mathbb{R}$ is $\langle \{\text{reals}\}; +, -, 0, \cdot, 1, <, >, |\cdot|, f\rangle$ for different choices of f.

(1) $f(x) = 2x$.

True. A winning strategy for $\exists$ is to look at the value of ε chosen by $\forall$ and then select $\delta = \varepsilon/2$. (To see that this is a winning strategy, observe that if this choice is made, then $0 < |x-1| < \delta$ implies $|f(x) - f(1)| = |2x-2| = 2|x-1| < 2\delta = \varepsilon$, so the statement is true.)

(2) $f(x) = x^2$.

True. A winning strategy for $\exists$ is to look at the value of ε chosen by $\forall$ and then select $\delta = \min\{1, \varepsilon/3\}$. (Why does this work?)

Switching the order of quantifiers.

For the next two problems, L is a language with exactly one binary predicate symbol P.

(3) Is there an L-structure where $\exists x\ \forall y\ P(x,y)$ is true while $\forall y\ \exists x\ P(x,y)$ is false?

No. If $\exists x\ \forall y\ P(x,y)$ is true in $\mathbb{A}$, then $\exists$ has a winning strategy for the associated quantifier game. That is, $\exists$ can choose some special value $a_0 \in A$ to assign to x so that $P(a_0, b)$ is true for any value b that $\forall$ assigns to y. But now $\exists$ can use the same strategy to win the game associated to $\forall y\ \exists x\ P(x,y)$: whichever value b that $\forall$ assigns to y, $\exists$ can assign a_0 to x to make $P(a_0, b)$ true.

(4) Is there an L-structure where $\forall y\ \exists x\ P(x,y)$ is true while $\exists x\ \forall y\ P(x,y)$ is false?

Yes. Let the structure $\mathbb{A} = \langle A; P\rangle$ be the one where $A = \mathbb{R}$ = real numbers and $P(x,y) = x < y$. Then $\forall y\ \exists x\ P(x,y)$ asserts that every real dominates a smaller real (which is true), while $\exists x\ \forall y\ P(x,y)$ asserts that there is a real dominated by all reals (which is false).

[1]Technically, we should not allow predicate symbols to appear in the quantifier part, but it is common to use the abbreviations $(\forall x > 0)\ \alpha(x)$ and $(\exists y > 0)\ \beta(y)$ to mean $\forall x\ ((x > 0) \rightarrow \alpha(x))$ and $\exists y\ ((y > 0) \wedge \beta(y))$.