Formulas for Distributions.

How many ways can one distribute k balls to n distinct boxes?

Assumption 1. No bound on the number of balls per box.

Each box must get a ball?

Balls distinct?

	Yes	No
Yes	$n!S(k,n)$	n^k
No	$\binom{k-1}{n-1}$	$\binom{n+k-1}{n-1}$

Assumption 2. Each box gets at most one ball (so $k \leq n$).

Each box must get a ball?

Balls distinct?

	Yes (so $k = n$)	No
Yes	$n!$	$(n)_k$
No	1	$\binom{n}{k}$

(1) How many ways are there to distribute 12 different books to 3 people? What if each person must get at least one book?

The problem concerns the distribution of 12 distinct books (= balls) to 3 distinct recipients (= boxes) where there is no bound on the number of balls ber box. In the first part some boxes may receive no balls: 3^{12} distributions. If every box gets a ball: $3!S(12,3)$ distributions.

(2) How many ways are there to distribute 12 identical textbooks to three shelves? How many ways to distribute 12 different books to three shelves?

There are $\binom{3+12-1}{3-1} = \binom{14}{2}$ ways distribute 12 identical books to three distinct shelves. If the books are distinct, then first put markers on the shelves to decided how many books go on each shelf, then order the books, then put the books on the markers in the prescribed order. As in the first part, there are $\binom{14}{2}$ ways to distribute 12 identical markers to 3 shelves. There are 12! ways to order the books. Once ordered, there is one way to put them on the markers. This yields $12!\binom{14}{2}$ ways to distribute books.

(3) How many 5 digit numbers have their digits in increasing or decreasing order? How many have their digits in nondecreasing or nonincreasing order? (If $n = abcde$, then the digits are in increasing order if $a < b < c < d < e$ and in nondecreasing order if $a \le b \le c \le d \le e$.)

The number of 5 digit numbers with digits decreasing is $\binom{10}{5}$, since once the subset of 5 digits have been chosen from the possible 10 digits there is exactly one way for them to be written in decreasing order. The same argument holds for 5 digit numbers with increasing digits, except we do not allow 0 to be a leading digit so we only choose our 5 digits from the 9 possible nonzero digits: $\binom{9}{5}$ ways. Thus, the answer to the first part is $\binom{10}{5} + \binom{9}{5}$ ways.

The second part can be done like the first part, except that instead of picking a subset of 5 distinct digits we want to pick a multiset of 5 not necessarily distinct digits. E.g., we are allowed to pick the multiset $\{3, 3, 5, 7, 7\}$, which represents the number 33577, whose digits are nondecreasing.

Choosing a 5 element multiset from a set of 10 distinct digits may be viewed as distributing 5 identical balls (representing membership in the multiset) to 10 boxes (representing the 10 digits) in a way that each box may get zero or more balls. E.g., the multiset $\{3, 3, 5, 7, 7, \}$ corresponds to the distribution that puts 2 identical balls into box #3, 1 ball in box #5, and 2 identical balls into box #7.

Combining the idea from the first part of the problem with the idea from the previous paragraph, we get that the number of 5 digit numbers with digits in nonincreasing order is $\binom{10+5-1}{10-1} - 1 = \binom{14}{9} - 1$. (The -1 is included to eliminate 00000 from the count. This is the only sequence of 5 nonincreasing digits that has a leading 0.)

Similarly, the number of 5 digit numbers with digits in nondecreasing order is $\binom{9+5-1}{9-1} = \binom{13}{8}$.

The full answer to the problem is given by inclusion/exclusion: the number counted in the previous paragraph plus the number counted in the paragraph before minus those counted in both paragraphs: $\binom{13}{8} + (\binom{14}{9} - 1) - 9 = \binom{13}{8} + \binom{14}{9} - 10$. (The -9 is to adjust for the fact that $11111, 22222, \ldots, 99999$ were counted twice.)

(4) How many positive integral solutions are there to the equation $x_1+x_2+x_3+x_4+x_5+x_6 = 100$? How many nonnegative integral solutions are there?

This may be viewed as the problem counting the number of distributions of 100 identical balls to 6 boxes labeled $x_1, \ldots, x_6$. The answer to the first part is $\binom{6+100-1}{6-1} = \binom{105}{5}$. If the solutions must be positive, then every box must get a ball, so the answer is $\binom{100-1}{6-1} = \binom{99}{5}$.

(5) How many ways are there to make 3 fruit baskets from 8 pineapples, 10 pomegranates, 6 coconuts and 20 figs if each basket must contain each kind of fruit?

Let the 3 baskets be boxes, and distribute the identical fruit one at a time. There are $\binom{8-1}{3-1} = \binom{7}{2}$ ways to distribute the pineapples, $\binom{10-1}{3-1} = \binom{9}{2}$ ways to distribute the pomegranates, $\binom{6-1}{3-1} = \binom{5}{2}$ ways to distribute the coconuts, and $\binom{20-1}{3-1} = \binom{19}{2}$ ways to distribute the figs. Thus, the number of ways to make 3 baskets is $\binom{7}{2}\binom{9}{2}\binom{5}{2}\binom{19}{2}$.