DISCRETE MATH
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. What is the definition of the italicized word or phrase?

(a) "E is an ***equivalence relation*** on the set A".

E is an ***equivalence relation*** on the set A if E is a binary relation on A that is reflexive, symmetric and transitive.

(b) "P is a ***partition*** of the set A".

P is a ***partition*** of the set A if $P = \{A_i \mid i \in I\}$ is a set of nonempty subsets of A such that (i) $\cup_{i\in I} A_i = A$ and (ii) if $A_i \cap A_j \neq \emptyset$, then $A_i = A_j$.

2. Give an example of:

(a) sets $A \in B \in C$ such that $A \not\subseteq B \subseteq C$.

$A = \{a\}$, $B = \{\{a\}\}$, and $C = \{\{\{a\}\}, \{a\}\}$.

(b) a function $F\colon \mathbb{R} \to \mathbb{R}$ that is onto but not 1-1.

Any polynomial of odd degree that has at least 2 distinct roots will work, for example $F(x) = x^2(x-1)$. This function is not 1-1 since $F(0) = F(1)$ and $0 \neq 1$. It is onto because $\lim_{x\to-\infty} F(x) = -\infty$, $\lim_{x\to+\infty} F(x) = +\infty$, and F is continuous (hence attains every value between $-\infty$ and $+\infty$).

3.

(a) What is the recursive definition of multiplication of natural numbers?

$$\begin{array}{lrl} \text{(IV)} & m \cdot 0 & := 0 \\ \text{(RR)} & m \cdot S(n) & := m \cdot n + m \end{array}$$

(b) Prove that $m \cdot 1 = 1 \cdot m = m$ for all $m \in \mathbb{N}$. (You may use any previously proved theorems that concern *addition*.)

First let's prove that $m \cdot 1 = m$ for all m.

$$\begin{array}{rll} m \cdot 1 & = m \cdot S(0) & \text{(Defn of 1)} \\ & = m \cdot 0 + m & ((\text{RR}), \cdot) \\ & = 0 + m & ((\text{IV}), \cdot) \\ & = m & \text{(0 is an additive unit, proved earlier)} \end{array}$$

Now let's prove by induction the statement "$1 \cdot m = m$".

(Basis of induction, $m = 0$)

$$1 \cdot 0 \quad = 0 \qquad\qquad ((\text{IV}), \cdot).$$

(Inductive step) Assume $1 \cdot m = m$ and show $1 \cdot S(m) = S(m)$

$$\begin{array}{rll} 1 \cdot S(m) & = (1 \cdot m) + 1 & ((\text{RR}), \cdot) \\ & = m + 1 & \text{(Inductive Hypothesis)} \\ & = S(m) & \text{(Theorem "}S(m) = m + 1\text{", from class).} \end{array}$$

4. Find $x, y \in \mathbb{Z}$ such that $1001x + 168y = 42$.

First we'll apply the Euclidean Algorithm.

$$\begin{aligned}
1001 &= 168 \cdot q_1 + r_1 && (1001, 168, \cdots) \\
&= 168 \cdot 5 + 161 && (1001, 168, 161, \cdots) \\
\\
168 &= 161 \cdot q_2 + r_2 && (1001, 168, 161, \cdots) \\
168 &= 161 \cdot 1 + 7 && (1001, 168, 161, 7, \cdots) \\
\\
161 &= 7 \cdot q_3 + r_3 && (1001, 168, 161, 7, \cdots) \\
161 &= 7 \cdot 23 + 0 && (1001, 168, 161, 7, 0).
\end{aligned}$$

Next we use back substitution.

$$\begin{aligned}
7 &= 168 - 161 \cdot 1 \\
&= 168 - (1001 - 168 \cdot 5) \cdot 1 \\
&= 168 \cdot 6 - 1001 \cdot 1 \\
&= 1001(-1) + 168(6),
\end{aligned}$$

so $1001(-1) + 168(6) = \gcd(1001, 168) = 7$. Multiplying this equation by 6 yields $1001(-6) + 168(36) = 42$, so we can take $x = -6$ and $y = 36$.

(All other solutions have the form $x = -6 + 24t$, $y = 36 - 143t$ for $t \in \mathbb{Z}$.)