Modern Algebra 2 (MATH 6140)
HW8, EXAMPLE (March 5, 2008)

Let $R = \mathbb{Z}$ be the ring of integers, a PID, and let M be the R-module presented by $\langle x, y, z \mid 2x + 3y + 5z = 0, y + z = 0, 2x + 4y + 6z = 0\rangle$. Let's work through the algorithm for determining the invariant factor form of M from this information.

The starting point is to present M by the exact sequence

$$\oplus^H R \xrightarrow{\ \varphi\ } \oplus^G R \longrightarrow M \longrightarrow 0,$$

where $G = (x, y, z)$, $H = (h_1, h_2, h_3)$ are ordered bases for the free modules in this sequence, and φ is determined by $h_1 \mapsto 2x + 3y + 5z, h_2 \mapsto y + z, h_3 \mapsto 2x + 4y + 6z$. The goal of the algorithm is to diagonalize the relations matrix ${}_G[\varphi]_H$. On the next page I record all information in five columns: the current set of generators for $\oplus^G R$, the row operation P used compute the next matrix, the current form of the relations matrix $[\varphi]$, the column operation Q used compute the next matrix, and the current set of generators for $\oplus^H R$. Here it should be noted that when using a column operation to change the generators of $\oplus^H R$ from a basis H_i to a basis H_{i+1} we use a matrix Q, that is the matrix for the transformation ${}_{H_i}[\mathrm{id}]_{H_{i+1}}$. Hence the columns of Q represent the elements of H_{i+1} written in terms of H_i. That is, one can determine H_{i+1} directly from the columns of Q. On the other hand, when using a row operation to change the generators of $\oplus^G R$ from a basis G_i to a basis G_{i+1} we use a matrix P that is the matrix for the transformation ${}_{G_{i+1}}[\mathrm{id}]_{G_i}$. This means that we use the columns of P^{-1} to determine G_{i+1} from G_i.

You should examine the next page now to see if you understand and agree with all the steps.

The final results should be interpreted as follows.

(1) $\bar{G} = (\bar{x}, \bar{y}, \bar{z}) := (2x + 3y + 5z, -x - y - 2z, z)$ is a new ordered basis for $\oplus^G R$.
(2) $\bar{H} = (\bar{h}_1, \bar{h}_2, \bar{h}_3) := (h_1, h_2 - h_1, h_3 - h_1 - h_2)$ is a new ordered basis for $\oplus^H R$.
(3) The (original) homomorphism φ satisfies $\varphi(\bar{h}_1) = \bar{x}$, $\varphi(\bar{h}_2) = 2\bar{y}$, $\varphi(\bar{h}_3) = 0\bar{z}$.

G_i	P_i	$[\varphi]$	Q_i	H_i
x, y, z		$\begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 1 & 6 \end{bmatrix}$		h_1, h_2, h_3
	$\begin{bmatrix} -1 & 1 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$			
$2x + 3y, -x - y, z$		$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 5 & 1 & 6 \end{bmatrix}$		h_1, h_2, h_3
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \times$			
$2x + 3y + 5z, -x - y, z$		$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix}$		h_1, h_2, h_3
			$\times \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$2x + 3y + 5z, -x - y, z$		$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix}$		$h_1, h_2 - h_1, h_3$
			$\times \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$2x + 3y + 5z, -x - y, z$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix}$		$h_1, h_2 - h_1, h_3 - 2h_1$
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \times$			
$2x + 3y + 5z, -x - y - 2z, z$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$		$h_1, h_2 - h_1, h_3 - 2h_1$
			$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	
$2x + 3y + 5z, -x - y - 2z, z$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		$h_1, h_2 - h_1, h_3 - h_1 - h_2$

So $M \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})/(1\mathbb{Z} \oplus 2\mathbb{Z} \oplus 0\mathbb{Z}) \cong (\mathbb{Z}/1\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/0\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$.