Practice Problems.

(1) (Berkeley, Prelim, S00) Are the 4×4 matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

similar? Explain your reasoning.

(2) (Berkeley, Prelim, S00) Let A be an $n \times n$ matrix over $\mathbb{F}$ whose minimal polynomial has degree k. Prove that if $\lambda \in \mathbb{F}$ is not an eigenvalue of A, then there is a polynomial $p(x) \in \mathbb{F}[x]$ of degree $< k$ such that $p(A) = (A - \lambda I)^{-1}$.

(3) (Berkeley, Prelim, S99) Let M be an $n \times n$ matrix with entries in the polynomial ring $\mathbb{R}[t]$ such that

$$M^3 = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}.$$

Let N be the matrix with real entries obtained from M by substituting 0 in for t. Prove that N is similar to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(4) (Berkeley, Prelim, S98) Let A be an $N \times N$ complex matrix whose eigenvalues sum to 0. Show that A is similar to a matrix with all 0's along the main diagonal.

(5) (Berkeley, Prelim, F97) Prove that if A is a 2×2 matrix over the integers such that $A^n = I$ for some strictly positive integer n, then $A^{12} = I$.

(6) (Berkeley, Prelim, F94) Write down a list of 5×5 complex matrices, as long as possible, with the following properties:
(a) The characteristic polynomial of each matrix in the list is x^5;
(b) The minimal polynomial of each matrix in the list is x^3;
(c) No two matrices in the list are similar.

(7) (Dartmouth, Quals, 96) If G is a finite abelian group, and $G \otimes \mathbb{Z}_p = 0$ for all primes p, show that $G = 0$. Does the result remain true if G is infinite?

(8) (Dartmouth, Quals, 96) Show that if $T : V \to V$ is a linear transformation over a field of characteristic $\neq 2$, and $T^2 = I$, then T is diagonalizable.

(9) (Missouri at Columbia, Quals, F00) Exhibit all 6×6 nilpotent complex matrices, up to similarity. (A matrix is *nilpotent* if some power is 0.)

(10) (Nebraska, Quals, S00) Let A be an 8×8 real matrix with determinant 16 and minimal polynomial $(x-2)^2(x^2+1)$. What are the possible rational canonical forms and Jordan canonical forms for A?

(11) (Penn State, Quals, F99) Determine the structure of the abelian group given by generators x, y, and z and the relations $7x+2y+3z = 0$, $21x+8y+9z = 0$, and $5x - 4y + 3z = 0$.

(12) (Temple, Quals, F95) Let $T : V \to V$ be an endomorphism. Suppose that for some $k \geq 1$ and some $v \in V$ we have $T^k(v) = 0$ and $T^{k-1}(v) \neq 0$. Show that x^k divides the minimal polynomial of T.

(13) (Columbia, Quals, F95) Let M be a Noetherian $\mathbf{R}$-module, and let $\varphi : M \to M$ be a surjective homomorphism. Show that φ is an isomorphism.

(14) (Johns Hopkins, Quals, F00) Let $\mathbf{R} = \mathbb{Q}[x]/(x^7 - 1)$. Give an example of an $\mathbf{R}$-module that is not free but is a direct summand of a free $\mathbf{R}$-module.

(15) (Rochester, Prelims, S95) Show that an Artinian ring has only finitely many maximal ideals.

(16) (Oklahoma, Quals, S00) Show that if $\mathbf{R}[x]$ is a PID, then $\mathbf{R}$ is a field.