

Modern Algebra 2 (MATH 6140)
Test #1 (February 13, 2008)

Name: _____

Do **two** of the problems. In order to receive full credit your answer must be **complete, legible and correct**.

1. Let \mathcal{V} be a variety, and suppose that $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ are presented by $\langle G \mid R \rangle$ and $\langle H \mid S \rangle$ relative to \mathcal{V} , respectively. Show that if G and H are disjoint, then $\langle G \cup H \mid R \cup S \rangle$ is a presentation of the coproduct of \mathbf{A} and \mathbf{B} relative to \mathcal{V} . (Hint: the first step should be to prove the existence of the coprojection homomorphisms.)

Notation. Let $\mathbf{C} = \langle G \cup H \mid R \cup S \rangle$. Also, since $\langle G \mid R \rangle = \mathbf{F}_{\mathcal{V}}(G)/\theta(R)$, let's agree to write \bar{g} for $g/\theta(R)$ if $g \in G$. Use this 'bar' notation for any presentation to denote the congruence class of a generator.

Solution. Since the generators G in \mathbf{C} satisfy the relations in R , it follows from the universal property of presentations, applied to $\langle G \mid R \rangle$, that the function $G \rightarrow G \cup H \rightarrow \mathbf{C}: g \mapsto g \mapsto \bar{g}$ extends to a homomorphism $i_1: \mathbf{A} \rightarrow \mathbf{C}$. There is a similar homomorphism $i_2: \mathbf{B} \rightarrow \mathbf{C}$. To show that $\langle \mathbf{C}; i_1, i_2 \rangle$ is a coproduct of \mathbf{A} in \mathbf{B} , consider $\mathbf{D} \in \mathcal{V}$ and homomorphisms $\varphi_1: \mathbf{A} \rightarrow \mathbf{D}$ and $\varphi_2: \mathbf{B} \rightarrow \mathbf{D}$. The set $\varphi_1(G)$ satisfies the relations $\varphi_1(R)$ in \mathbf{D} , since φ_1 is a homomorphism, and for the same reason the set $\varphi_2(H)$ satisfies the relations $\varphi_2(S)$ in \mathbf{D} . Hence the function $\varphi_1|_G \cup \varphi_2|_H$ maps $G \cup H$ to elements of \mathbf{D} that satisfy the relations $\varphi_1(R) \cup \varphi_2(S)$. By the universal property of presentations, $\varphi_1|_G \cup \varphi_2|_H$ extends to a unique homomorphism $\varphi_1 \sqcup \varphi_2: \mathbf{C} \rightarrow \mathbf{D}$. Now, $(\varphi_1 \sqcup \varphi_2) \circ i_1$ and φ_1 both agree on G , which is a generating set for \mathbf{A} , so $(\varphi_1 \sqcup \varphi_2) \circ i_1 = \varphi_1$. Similarly, $(\varphi_1 \sqcup \varphi_2) \circ i_2 = \varphi_2$. This establishes that $\langle \mathbf{C}; i_1, i_2 \rangle$ satisfies the universal property for the coproduct of \mathbf{A} and \mathbf{B} .

2. Let R be a commutative ring. Show that if M and N are finitely generated (cyclic) R -modules, then so is $M \otimes_R N$. Now suppose that R, S and T are not-necessarily-commutative rings, that M is finitely generated (cyclic) as an (R, S) -bimodule, N is finitely generated (cyclic) as an (S, T) -bimodule. Must the $M \otimes_S N$ be finitely generated (cyclic) as an (R, T) -bimodule?

Solution. If M is generated as an R -module by G and N is generated by H , then it follows from the bilinearity of \otimes_R and the fact that the left and right actions of R on M and N are the same that $M \otimes_R N$ is generated as an R -module by $\{g \otimes h \mid (g, h) \in G \times H\}$. This proves the first assertion.

The second assertion is false. If $M = {}_{\mathbb{Z}}\mathbb{Q}_{\mathbb{Q}}$ and $N = {}_{\mathbb{Q}}\mathbb{Q}_{\mathbb{Z}}$, then both are 1-generated as bimodules, but $M \otimes_{\mathbb{Q}} N = \mathbb{Q}$ (as an abelian group/ (\mathbb{Z}, \mathbb{Z}) -bimodule), and \mathbb{Q} is not finitely generated as an abelian group/ (\mathbb{Z}, \mathbb{Z}) -bimodule.

3. Which rings R have the property that all R -modules are free? (Hint: Make use of a simple module, if possible.)

Solution. Zero rings have the property, because every module over a zero ring is zero, hence free with zero generators. Division rings have the property because any module over a division ring is a vector space, and from linear algebra we know that every vector space has a basis. No other ring has the property.

To see this, note that, if a simple R -module S is free, then it must be free over a 1-element set $X = \{x\}$, since minimal generating sets of simple modules have size 1. If $R \neq 0$, then R has a simple module, and it is isomorphic to R/M where $M = \text{Ann}(\{x\})$ is a maximal left ideal of R . Now if $m \in M - \{0\}$, then $mx = 0$ is a nontrivial relation satisfied by X , contradicting freeness unless $M = \{0\}$. But if $M = \{0\}$ for some maximal left ideal of R , then $L = \{0\}$ for all proper left ideals of R , which makes R a division ring. (To see this, choose any $a \in R - \{0\}$. $L = Ra$ is not zero, hence not proper, hence equals R , hence $ra = 1$ for some $r \in R$. This proves that every nonzero $a \in R$ has a left inverse, hence every element of R has a 2-sided inverse.)