Modern Algebra 2 (MATH 6140)
HANDOUT 5 (March 3, 2008)

CHAIN CONDITIONS, MODULARITY

Definition 1. (Terminology for ordered sets.) Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set.

(1) $\mathbf{P}$ is *well-founded* if every nonempty subset has a minimal element. (FYI: $\mathbf{P}$ is *well-ordered* if it is well-founded and linearly ordered.)
(2) $\mathbf{P}$ satisfies the *descending chain condition* (DCC) if it contains no strictly decreasing ω-indexed chain, $p_0 > p_1 > p_2 > \cdots$. The dual condition is the *ascending chain condition* (ACC).
(3) A subset $D \subseteq P$ is *down-directed* if whenever $a, b \in D$ there is a $c \in D$ such that $a \geq c$ and $b \geq c$. The dual property is *up-directed.*

Definition 2. (Terminology for lattices.) Let $\mathbf{L} = \langle L; \vee, \wedge \rangle$.

(1) $\mathbf{L}$ is *complete* if any (possibly empty) subset $Z \subseteq L$ has a least upper bound, $\bigvee Z$. (FYI: This concept should be called $\vee$-*complete*, but it can be proved to be equivalent to $\wedge$-complete, so we just say "complete".)
(2) An element $c \in L$ is *compact* if whenever $c \leq \bigvee Z$ for some $Z \subseteq L$, then there is a finite subset $Z_0 \subseteq Z$ such that $c \leq \bigvee Z_0$.
(3) A complete lattice is *compactly generated* if every element is the least upper bound of a family of compact elements. A compactly generated complete lattice is called an *algebraic lattice.*

Definition 3. (Terminology for closure operators.) Let $\mathsf{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator on X.

(1) cl is *algebraic* if $U \subseteq X$ is closed iff U contains the closure of each of its finite subsets.
(2) A closed set $U \subseteq X$ is *finitely generated* if $U = \mathsf{cl}(U_0)$ for some finite set $U_0 \subseteq X$.

Discussion. Well-founded sets are important in mathematics because they are the ordered sets over which induction is possible. The dual condition is closely linked to Zorn's Lemma. These two properties are characterized by DCC and ACC, respectively. In algebra, these order-theoretic properties are natural finiteness hypotheses on congruence lattices, which are typical examples of algebraic lattices.

Theorems. Theorem (E) is the important one. Theorems (A)–(C) are for background information only, hence are labeled [FYI]. Theorem (D) is used in the proof of (E).

(A) [FYI] A lattice is complete iff it is isomorphic to the lattice of closed sets of some closure operator.
(B) [FYI] The following conditions are equivalent for any lattice $\mathbf{L}$.
  (1) $\mathbf{L}$ is algebraic.
  (2) $\mathbf{L}$ is isomorphic to the lattice of closed sets of some algebraic closure operator.
  (3) $\mathbf{L}$ is isomorphic to the subalgebra lattice of some algebra.
  (4) $\mathbf{L}$ is isomorphic to the congruence lattice of some algebra.

(C) [FYI] A closure operator is algebraic iff the set-theoretic union of an up-directed family of closed sets is closed.

(D) If $\mathbf{L}$ is the lattice of closed sets of some algebraic closure operator, then an element $c \in L$ is compact iff it is finitely generated.

(E) The following conditions are equivalent for any ordered set $\mathbf{P}$.

(1) $\mathbf{P}$ satisfies the ACC.

(2) $\mathbf{P}$ is dually well-founded. (I.e., every nonempty subset of $\mathbf{P}$ has a maximal element.)

If $\mathbf{P}$ is a complete lattice, then these conditions are equivalent to:

(3) Every element of $\mathbf{P}$ is compact.

If $\mathbf{P}$ is the lattice of closed sets of some algebraic closure operator, then these conditions are equivalent to:

(4) Every element of $\mathbf{P}$ is finitely generated.

(F) Similarly, the duals of (E)(1)–(3) are equivalent for complete lattices.

Definition 4. A module is *Noetherian* if its submodule lattice satisfies ACC. (Dually, *Artinian.*) A ring $\mathbf{R}$ is *left Noetherian* if ${}_{\mathbf{R}}\mathbf{R}$ is a Noetherian $\mathbf{R}$-module. (Dually, *left Artinian.*)

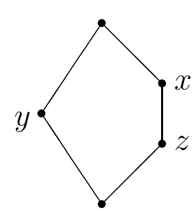
From Theorem (E) it follows that an $\mathbf{R}$-module M is Noetherian iff every submodule is finitely generated, and that $\mathbf{R}$ is left Noetherian if every left ideal of $\mathbf{R}$ is finitely generated as a left ideal.

Modularity.

Theorem 5. *(Dedekind) The following are equivalent for a lattice* $\mathbf{L}$.

(1) $\mathbf{L}$ *is* modular, *i.e. it satisfies* $\forall x, y, z((x \wedge (y \vee (x \wedge z))) = ((x \wedge y) \vee (x \wedge z)))$.

(2) $\mathbf{L}$ *has no sublattice isomorphic to the pentagon,* $\mathbf{N}_5$*:*



(3) *(Dedekind's Transposition Principle) If* $a, b \in L$*, then the function* $\rho_{\vee b}\colon [a \wedge b, a] \to [b, a \vee b]\colon x \mapsto x \vee b$ *is a lattice isomorphism whose inverse is* $\lambda_{a\wedge}\colon [a \wedge b, a \vee b] \to [a \wedge b, a]\colon y \mapsto a \wedge y$.

Theorem 6. *If a modular lattice* $\mathbf{L}$ *has an element* a *such that the intervals* $[0, a]$ *and* $[a, 1]$ *satisfy the ACC, then* $\mathbf{L}$ *satisfies the ACC. Same for DCC.*

Recall from the solution to HW1.2,3 that the congruence lattice of any algebra with underlying group structure satisfies Theorem 5 (2). Hence we get:

Theorem 7. *The submodule lattice of a module is modular.*

Corollary 8. *In the exact sequence of modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

M is Noetherian iff both L and N are. (Same for Artinian.)

Corollary 9. *A finitely generated module over a left Noetherian ring is Noetherian. (Same for Artinian.)*

Corollary 10. *A finitely generated module over a left Noetherian ring is finitely presentable. (Same for left Artinian, by Example 11 (5) and the first part of this corollary.)*

Example 11. (Of Noetherian and Artinian modules and rings.)
(1) A modular lattice satisfies ACC and DCC iff it has finite height, which means that there is a finite n such that every chain has size $\leq n$. Hence a module is both Noetherian and Artinian iff its submodule lattice has finite height.
(2) The class of Noetherian $\mathbf{R}$-modules is closed under the formation of homomorphic images, submodules and finite products.
(3) A PID is left Noetherian, since every left ideal is finitely generated.
(4) An algebraic number ring is left Noetherian.
(5) The Hopkins-Levitski Theorem asserts that every left Artinian ring is left Noetherian.
(6) A finite dimensional algebra over a field is left Artinian.
(7) A matrix ring over a division ring is left Artinian.

Notes on proofs.
$\vee$-complete $=$ $\wedge$-complete: If $\mathbf{L}$ is $\vee$-complete, then it has least and largest elements, $\bigvee \emptyset = 0$ and $\bigvee L = 1$. If $Z \subseteq L$ is nonempty, then let $B = \{b \in L \mid \forall z \in Z(b \leq z)\}$ be the set of lower bounds for the elements of Z. Then $\bigvee B$ exists by $\vee$-completeness, and belongs to B, hence equals $\bigwedge Z$. This shows that $\bigwedge Z$ exists for every $Z \subseteq L$.
Theorem (A): If $\mathbf{L}$ is the lattice of closed sets of a closure operator cl on X, and $Z \subseteq L$, then each $z \in Z$ is a (closed) subset of X. Thus $\mathrm{cl}(\bigcup Z) = \bigvee Z$ is the least element of L above each $z \in Z$. This shows that L is complete. Conversely, suppose that $\mathbf{L}$ is complete. Let $X = L$ and define cl as follows: for each $U \subseteq X$ let $\mathsf{cl}(U) = \{u \in X \mid u \leq \bigvee U\}$. Then cl is a closure operator on $X = L$ for which the closed sets are exactly the intervals $[0, \ell]$, $\ell \in L$. The inclusion ordering on closed sets agrees with the ordering on $\mathbf{L}$, so $\mathbf{L}$ is isomorphic to the lattice of closed sets of a closure operator.
Theorem (B): (Sketch) [(2)$\Rightarrow$(1)] is similar to the first part of the proof of (A), but it is useful to prove (D) first. [(1)$\Rightarrow$(2)] is similar to the proof of the second part of (A), but take X tyo be the set of compact elements of L instead of L itself. For [(2)$\Rightarrow$(3)] and [(2)$\Rightarrow$(4)] one only need show that subalgebra generation and congruence generation are algebraic closure operators. For the first of these use Theorem 5 of Handout 2, and for the second prove an analogue of Theorem 5 for congruences and follow the same line of argument. [(3)$\Rightarrow$(2)] is moderately hard; it is known as Birkhoff's Theorem. [(4)$\Rightarrow$(2)] is very hard; it is known as the Grätzer-Schmidt Theorem.
Theorem (C): (Sketch) For the forward direction, note that any up-directed union of closed sets contains the closures of any of its finite subsets. For the backward direction, use the

fact that the closures of finite subsets of U form an up-directed set, then apply the definition of "algebraic".
Theorem (D): Assume that $c \in L$ is compact. Let $Z = \{\mathsf{cl}(U) \mid U \subseteq c,\ U \text{ finite}\}$. Then $Z \subseteq L$ and $c = \bigvee Z$, so there is a finite subset $Z_0 \subseteq Z$ such that $c \leq \bigvee Z_0 \leq \bigvee Z = c$, or just $\bigvee Z_0 = c$. If $Z_0 = \{\mathsf{cl}(U_1), \ldots, \mathsf{cl}(U_n)\}$, then $c = \bigvee Z_0 = \mathsf{cl}(\bigcup_{i=1}^{n} U_i)$, exhibiting c as the closure of a finite set. Conversely, assume that $c = \mathsf{cl}(U)$ is the closure of a finite set $U = \{u_1, \ldots, u_k\}$. If $Z \subseteq L$ is such that $c \leq \bigvee Z = \mathsf{cl}(\bigcup Z)$, then each $u_i \in c$ lies in $\mathsf{cl}(\bigcup Z)$, so (since cl is algebraic) it follows that for each i there is a finite subset $Z_i \subseteq Z$ such that $u_i \in \mathsf{cl}(\bigcup Z_i)$. Hence, if $Z_0 = \bigcup_{i=1}^{k} Z_i$, then Z_0 is a finite subset of Z such that $c \leq \bigvee Z_i$.
Theorem (E): For [(1)$\Rightarrow$(2)], if $\mathbf{P}$ satisfies ACC, then every nonempty subset of $\mathbf{P}$ is inductively ordered, so apply Zorn's Lemma to get (2). For [$\neg$(1) $\Rightarrow$ $\neg$(2)], if $C \subseteq P$ is a strictly increasing ω-chain, then C has no maximal element. For [(2)$\Rightarrow$(3)], assume that $c \in P$, $Z \subseteq P$, and $c \leq \bigvee Z$. The set $Q = \{\bigvee Z_0 \mid Z_0 \subseteq Z,\ Z_0 \text{ finite}\}$ contains the least element of $\mathbf{P}$, so it is nonempty. By (2) it has a maximal element, M, but since Q is up-directed M is even a maximum element. This forces both $M = \bigvee Z$ and $M = \bigvee Z_0$ for some finite subset $Z_0 \subseteq Z$. Thus $c \leq \bigvee Z = M = \bigvee Z_0$ for some finite $Z_0 \subseteq Z$. This proves that c is compact. For [$\neg$(1) $\Rightarrow$ $\neg$(3)], suppose that C is a strictly increasing ω-chain and that $c = \bigvee C$. Then $c \in P$ cannot be compact, since $c \leq \bigvee C$ but $c \not\leq \bigvee C_0$ for any finite subset $C_0 \subseteq C$. Finally, [(3) $\Leftrightarrow$ (4)] by Theorem (D).
Theorem 5: For [(1)$\Rightarrow$(2)], the modular law is universally quantified, hence is inherited by sublattices. Since $\mathbf{N}_5$ is not modular, it cannot occur as a sublattice of a modular lattice. For [$\neg$(1) $\Rightarrow$ $\neg$(2)], if $\mathbf{L}$ has elements a, b and c such that $a \wedge (b \vee (a \wedge c)) \neq (a \wedge b) \vee (a \wedge c)$, then the lefthand side of this equality must be strictly larger than the righthand side (since $a \wedge (b \vee (a \wedge c)) \geq (a \wedge b) \vee (a \wedge c)$ is derivable from the lattice laws). Now it can be shown that $x := a \wedge (b \vee (a \wedge c))$, $y := b$ and $z := (a \wedge b) \vee (a \wedge c)$ generate a pentagon. For [(1)$\Rightarrow$(3)], assume that $\mathbf{L}$ is modular and $a, b \in L$. From the modular law, we have $\lambda_{a\wedge}(\rho_{\vee b}(x)) = a \wedge (x \vee b) = a \wedge ((a \wedge x) \vee b) = (a \wedge x) \vee (a \wedge b) = x \vee (a \wedge b) = x$ whenever $a \wedge b \leq x \leq a$. Similarly, by the dual of the modular law,[1] if $\rho_{\vee b}(\lambda_{a\wedge}(y)) = (a \wedge y) \vee b = (a \wedge (y \vee b)) \vee b = (a \vee b) \wedge (y \vee b) = (a \vee b) \wedge y = y$ whenever $b \leq y \leq a \vee b$. Thus, $\lambda_{a\wedge}$ and $\rho_{\vee b}$ are order-preserving bijections between $[a \wedge b, a]$ and $[b, a \vee b]$, hence they are isomorphisms between these intervals. [$\neg$(2) $\Rightarrow$ $\neg$(3)] If $\mathbf{L}$ has a sublattice isomorphic to $\mathbf{N}_5$, labeled as in Theorem 5 (2), then $x \neq z$ but $\lambda_{y\wedge}(x) = \lambda_{y\wedge}(z)$, so $\lambda_{y\wedge}$ is not an isomorphism from $[z, y \vee z]$ to $[y \wedge z, y]$.
Theorem 6: Suppose otherwise that $b_0 < b_1 < \cdots$ is an infinite strictly ascending chain. Then $a \vee b_0 \leq a \vee b_1 \leq \cdots$ is an ascending chain above a, and $a \wedge b_0 \leq a \wedge b_1 \leq \cdots$ is an ascending chain below a. Since these chains lie in intervals with the ACC, only finitely many of their terms are distinct. Thus, there exists a k such that $b_k < b_{k+1}$, $a \vee b_k = a \vee b_{k+1}$, and $a \wedge b_k = a \wedge b_{k+1}$. This implies that the sublattice generated by $\{a, b_k, b_{k+1}\}$ is a pentagon, contradicting modularity.

[1]The modular law is equivalent to its dual, since the definition of "lattice" is self-dual, and $\mathbf{N}_5$ is self-dual.