Modern Algebra 2 (MATH 6140)
HANDOUT 3 (February 1, 2008)

CATEGORIES, FUNCTORS, UNIVERSAL ARROWS

A category is an algebraic model of a collection of mathematical structures (*objects*) equipped with structure preserving mappings (*morphisms*).

Definition 1. (Category) A *category* is a structure

$$\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$$

where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called *the identity of* A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$, $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
(4) If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, then $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.
(5) $\mathrm{dom}(\mathrm{id}_A) = \mathrm{cod}(\mathrm{id}_A) = A$.

We will say that a category is *small* if M is a set. (This forces O to be a set, too.)

Notation. We let $\mathrm{Hom}_{\mathcal{C}}(A, B)$ denote the class of $f \in M$ for which $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$. It is common to add to the definition of a category the assumption that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for all $A, B \in O$. (I will always assume this.)

Examples.

(1) If $\mathcal{V}$ is a variety or prevariety of algebras, then $\mathcal{V}$ may be thought of as a category whose objects are the algebras in $\mathcal{V}$ and whose morphisms are the homomorphisms between members of $\mathcal{V}$.
(2) If $\langle P; \leq \rangle$ is a partially ordered set, then the elements of P may be thought of as the objects of a category whose morphisms are the arrows $a \to b$ where $a \leq b$ in P.
(3) If $\langle M; \circ, 1 \rangle$ is a monoid, then M determines a one-object category

$$\mathcal{M} = \langle \{*\}, M; \circ, 1, \mathrm{dom}, \mathrm{cod} \rangle$$

where $\mathrm{dom}, \mathrm{cod} : M \to \{*\}$ are both the constant function.

Since the definition of a category is so close to that of an algebra, it is natural to try to compare categories with homomorphisms. These are called "covariant functors".

Definition 2. (Functor) A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. Precisely, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,
(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and
(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ is an antihomomorphism (a composition reversing mapping) from $\mathcal{C}$ to $\mathcal{D}$. That is, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, and

(1) $F(f \circ g) = F(g) \circ F(f)$,
(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
(3) $F(\mathrm{dom}(f)) = \mathrm{cod}(F(f))$, and
(4) $F(\mathrm{cod}(f)) = \mathrm{dom}(F(f))$.

Examples.

(1) Let $\mathcal{C}$ be any category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be any object in $\mathcal{C}$. The *covariant hom functor represented by* A is the covariant functor $F : \mathcal{C} \to \mathcal{SET}$ whose behavior on objects is $F(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$ and whose behavior on morphisms is $F(f) =$ left composition with f.
(2) The *contravariant hom functor represented by* A is the contravariant functor $F : \mathcal{C} \to \mathcal{SET}$ whose behavior on objects is $F(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$ and whose behavior on morphisms is $F(f) =$ right composition with f.
(3) If we consider posets P and Q to be categories (as earlier), then a function from P to Q is a covariant functor if it is order-preserving and is a contravariant functor if it is order-reversing.
(4) If we consider monoids M and N as one-object categories, then a covariant functor from M to N is a monoid homomorphism.
(5) A functor is *faithful* if it is injective on both objects and morphisms. A *concrete category* is a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and $U\colon \mathcal{C} \to \mathcal{SET}$ is a faithful functor to the category of sets. U provides a way to view the objects of $\mathcal{C}$ as having underlying sets and morphisms as being set-maps.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, and X is an object of $\mathcal{D}$, then it may happen that there is a "best projection" of X onto the image of F. This best projection consists of an object A of $\mathcal{C}$ and a morphism $\pi\colon X \to F(A)$ projecting X to the image of F, which satisfy a property saying that (A, π) is best.

Definition 3. (Universal morphism) Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and an object X of $\mathcal{D}$, a *universal morphism* from X to F is a pair (A, π) such that $\pi\colon X \to F(A)$, and whenever $\rho\colon X \to F(B)$ there is a unique $\sigma\colon A \to B$ such that $\rho = F(\sigma) \circ \pi$.

Similarly, a *universal morphism* from F to X is a pair (A, π) such that $\pi\colon F(A) \to X$, and whenever $\rho\colon F(B) \to X$ there is a unique $\sigma\colon B \to A$ such that $\rho = \pi \circ F(\sigma)$.

Similar definitions can be made for contravariant functors.

The statement that some universal property holds is the statement that a certain pair (A, π) is a universal morphism.

Exercise.

(1) Show that universal morphisms are unique up to unique isomorphism.

Examples.

(1) Let $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the diagonal functor ($A \mapsto (A, A)$ and $\varphi \mapsto (\varphi, \varphi)$). A universal morphism from Δ to an object (B, C) of $\mathcal{C} \times \mathcal{C}$ is a pair $(P, (\pi_1, \pi_2))$ where $(\pi_1, \pi_2)\colon (P, P) \to (B, C)$ is universal; this means exactly that P is a product of B and C and that π_1 and π_2 are the projection maps of the product. The universal property of products is the statement that $(P, (\pi_1, \pi_2))$ is a universal morphism.

(2) A universal morphism from (B, C) to Δ is a pair $(D, (i_1, i_2))$ where D is the coproduct of B and C and i_1 and i_2 are the coprojections. The universal property of coproducts is the statement that $(D, (i_1, i_2))$ is a universal morphism.

(3) If $\mathcal{P}$ is a prevariety, then a universal morphism from a set X to the natural forgetful functor $U\colon \mathcal{P} \to \mathcal{SET}$ is a pair $(F(X), \subseteq)$ where $F(X)$ is the underlying set of a free algebra over X and $\subseteq\colon X \to F(X)$ is inclusion of generators. The universal property of free algebras is the statement that $(F(X), \subseteq)$ is a universal morphism.

(4) Let $\mathcal{V}_{\mathbb{F}}$ be the category of $\mathbb{F}$-vector spaces. For objects U and V if this category, let $H\colon \mathcal{V}_{\mathbb{F}} \to \mathcal{SET}$ be the functor that assigns to W the set of bilinear maps $\mathrm{Bilin}(U \times V, W)$. There is a universal morphism from the 1-point set to H: $(U \otimes V, \beta)$ where $\beta(*)$ is the insertion of simple tensors $\mu\colon U \times V \to U \otimes V$.