Modern Algebra 2 (MATH 6140)
HW7, EXERCISE 2 (March 5, 2008)

Exercise 2. Let S is a finite subset of the unit interval that includes both endpoints. Assume that every element in S other than 0, 1 is the average of two other elements of S. Show that every element of S is rational. (Hint: Use a slight refinement of Minkowski's Criterion.)

Solution. Suppose that $S = \{0, a_1, \dots, a_n, 1\}$ with $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$. We are told that each a_i is an average of two other elements, so there exist a_j and a_k with $j \neq i \neq k$ such that $a_i = (a_j + a_k)/2$. For this to happen, it must be that a_i lies between a_j and a_k. Thus, for every i there exist $j < i < k$ such that $-a_j + 2a_i - a_k = 0$. This is a linear equation satisfied by the a's, and we have n such equations. In matrix form this may be written:

$$(1) \qquad \left[\begin{array}{r|rrrcrr|r} -1 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & \cdots & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & 0 & \cdots & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 2 & -1 \end{array}\right] \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

I've added extra vertical lines to the matrix to suggest cutting off the first and last columns. If these columns are cut off, then the resulting matrix satisfies the hypotheses of the next lemma.

Lemma 1. *If A is an $n \times n$ real matrix such that*

(i) *all diagonal entries are positive,*
(ii) *all off-diagonal entries are nonpositive,*
(iii) *all row sums are nonnegative, and*
(iv) *each row with row sum zero has at least one nonzero entry left of the diagonal,*

then A is nonsingular.

Proof. Suppose otherwise that some $\mathbf{x} = [x_1, \dots, x_n]^t \neq \mathbf{0}$ is in the nullspace of A. Choose i minimal so that $|x_i| = \max_j(|x_j|)$. This maximum is nonzero, since $\mathbf{x} \neq \mathbf{0}$. Multiplying the i-th row of A times $\mathbf{x}$ yields $\sum_{j=1}^n a_{ij}x_j = 0$, so $a_{ii}x_i = -\sum_{j\neq i} a_{ij}x_j$. Applying the triangle inequality to this equality and using conditions (i) and (ii) yields the first inequality in

$$(2) \qquad a_{ii}|x_i| \leq \sum_{j\neq i} -a_{ij}|x_j| \leq \sum_{j\neq i} -a_{ij}|x_i|.$$

The second inequality follows from the choice of x_i. Inequality (2) implies

$$(3) \qquad \left(\sum_{j=1}^n a_{ij}\right)|x_i| \leq 0.$$

We chose i so that $|x_i| > 0$, and by condition (iii) we have $\sum_{j=1}^n a_{ij} \geq 0$, so (3) forces $\sum_{j=1}^n a_{ij} = 0$. But this means that the left and right sides of (2) are equal, so the middle is also equal, and this forces $|x_j| = |x_i|$ for all j for which $a_{ij} \neq 0$. Now by condition (iv) there is a $j < i$ for which $a_{ij} \neq 0$, so $|x_j| = |x_i|$ for some $j < i$. This contradicts the minimality assumption made in the choice of i. $\square$

The lemma shows that the $n \times (n+2)$-matrix from (1) has an $n \times n$ submatrix of rank n, so the matrix itself must be of rank n. By the rank + nullity theorem, the kernel of this matrix has nullspace of rank 2. One vector in the nullspace is $\mathbf{a} = [0, a_1, \ldots, a_n, 1]^t$, while another vector in the nullspace is $\mathbf{1} = [1, 1, \ldots, 1, 1]^t$. The set $\{\mathbf{a}, \mathbf{1}\}$ is independent, hence a basis for the nullspace of the matrix in (1).

On the other hand, a basis for the nullspace of the matrix from (1) can be obtained by Gaussian elimination, and this method produces a basis consisting of two rational vectors, say $\{\mathbf{u}, \mathbf{v}\}$. (Here we are using the fact that the matrix in (1) has rational entries.)

Thus, $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{a}, \mathbf{1}\}$ are bases for the same subspace of $\mathbb{R}^{n+2}$. This means we can express $\mathbf{u} = p\mathbf{a} + q\mathbf{1}$ and $\mathbf{v} = r\mathbf{a} + s\mathbf{1}$. Writing these equalities in matrix form we find that

$$\begin{bmatrix} \mathbf{a} & \mathbf{1} \end{bmatrix} \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}. \tag{4}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are independent, the 2×2 matrix in the middle of line (4) is invertible. If we consider what (4) says in the first and last rows of $\begin{bmatrix} \mathbf{a} & \mathbf{1} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ we find that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{bmatrix} u_0 & v_0 \\ u_{n+1} & v_{n+1} \end{bmatrix}.$$

Left-multiplying by the inverse of the left-most matrix to solve for p, q, r, and s shows that they are rational. This, and

$$\begin{bmatrix} \mathbf{a} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} p & r \\ q & s \end{bmatrix}^{-1},$$

which follows from (4), implies that $\begin{bmatrix} \mathbf{a} & \mathbf{1} \end{bmatrix}$ is rational. Hence $\mathbf{a}$ is rational.