Modern Algebra 2 (MATH 6140)
HW #8 (Due March 12, 2008)

12.1.2. Let M be a module over an integral domain R.

(a) Suppose X is a maximal independent subset of M and $N = \langle X \rangle$. Show that N is free of rank $|X|$ and M/N is torsion.

Solution: N has X as an independent generating set, so N is free of rank $|X|$. If $y \in N$, then $\bar{y} = y/N = \bar{0}$ is torsion in M/N. If $y \in M - N = M - \langle X \rangle$, then $y \notin X$, so $X \cup \{y\}$ is not independent. Say $ry + \sum r_i x_i = 0$ is a nontrivial dependence relation. It must be that $r \neq 0$, since otherwise X satisfies a nontrivial dependence relation. Factoring modulo N we obtain $r\bar{y} = \bar{0}$, showing that $\bar{y}$ is torsion. $\bar{y} \in M/N$ was arbitrary, so M/N is torsion.

(b) Suppose conversely that of $N \leq M$, N is free of rank n, and M/N is torsion, then M has rank n.

Solution: Since $M \supseteq N$ and N has n linearly independent elements, $\mathrm{rk}(M) \geq n$. For the reverse inclusion, suppose that $y_1, \ldots, y_k$ are linearly independent elements of M. Since M/N is torsion, there is an $s \neq 0$ such that $sy_i \in N$ for all i. But now $sy_1, \ldots, sy_k$ are linearly independent elements of N. By Proposition 3, $k \leq n$, so $\mathrm{rk}(M) \leq n$.

2. Let R be an integral domain and let Q be its field of fractions.

(a) Show that the functor of tensoring with Q is an exact functor from R-modules to R-modules. (Q is *flat*.)

Solution:
Stage 1: Cite Theorem 39 to show that tensoring with any module is a right exact functor. Thus $Q \otimes \_$ is exact iff it maps monomorphims to monomorphisms.

Stage 2: Show that any element of $Q \otimes N$ reduces to a simple tensor. (Find a common denominator.)

Stage 3: Follow the hints in Exercise 10.4.8 to show that the function that maps $(a/b, n)$ to (b, an) induces an isomorphism of $Q \otimes N$ onto $(R^{\times})^{-1}N$. This gives a test for when a simple tensor $(a/b) \otimes n$ is zero, namely $(a/b) \otimes n = 0$ iff $(b, an) \sim (1, 0)$ iff $\exists r \in R^{\times}(r(an) = 0)$ iff an is torsion.

Stage 4: Use the criterion from Stage 3 for when a simple tensor is zero to show that $Q \otimes \_$ preserves monomorphisms. Namely, if $0 \to N \xrightarrow{f} M$ and $(a/b) \otimes n$ belongs to the kernel of $(1 \otimes f)\colon Q \otimes N \to Q \otimes M$, then $(1 \otimes f)((a/b) \otimes n) = (a/b) \otimes f(n) = 0$, which means that $af(n) = f(an)$ is torsion. But since f is a monomorphism, this means that an is torsion, that is $(a/b) \otimes n = 0$. Hence $1 \otimes f$ is a monomorphism.

(b) Show that the Q-dimension of $Q \otimes M$ is $\mathrm{rk}(M)$.

Solution: Suppose that X is a maximal independent subset of M, and $N = \langle X \rangle$. Then N is free of rank $|X|$, so $N \cong \oplus^{|X|} R$. Tensoring

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

over R with the (Q,R)-bimodule Q, and using that $Q \otimes N \cong Q \otimes (\oplus^{|X|} R) \cong \oplus^{|X|}(Q \otimes R) \cong \oplus^{|X|} Q$ and $Q \otimes M/N = 0$, we obtain an exact sequence

$$0 \longrightarrow \oplus^{|X|} Q \longrightarrow Q \otimes M \longrightarrow 0 \longrightarrow 0,$$

which means that $Q \otimes M \cong \oplus^{|X|} Q$ as Q-modules. This proves that $\dim_Q(Q \otimes M) = |X| = \mathrm{rk}(M)$.

3. Let M be the module presented by generators $G = \{x, y, z\}$ and relations $R = \{2(1+i)x - 3(1-i)y + (1+i)z = 0, 4(1-i)x + 7(1+i)y + (1-i)z = 0, 6ix - 10y + 2iz = 0\}$ over the PID $Z[i]$.

(a) Write M in invariant factor form.

Solution: Following the algorithm from the notes, we left multiply by $P_1 = \begin{bmatrix} -1 & -i & 0 \\ -3i & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, and $P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{bmatrix}$, then right multiply by $Q_1 = \begin{bmatrix} 1 & 3i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 1 & 0 & -2(1+i) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1+i \end{bmatrix}$. We obtain the equivalent matrix $\begin{bmatrix} 1+i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $M \cong \mathbb{Z}[i]/(1+i) \oplus Z[i]/(2) \oplus Z[i]$.

(b) Express the new generators in terms of the original generators.

Solution: Inverting the P matrices from part (a), and examining their effect on the generators, we find that the new generators are $2x + 3iy + z, ix - y + iz, z$.