Practice Sheet F

1. Let $f(x)$ be a continuous function on$[0,1]$ and $f(0) = f(1) = 0$. Let $0 < \alpha < 1$. Show that there exist x, y in $[0,1]$ such that $f(x) = f(y)$ and $x - y$ is either α or $1 - \alpha$.

2. A circle with the center $(a, 1/a)$ intersects the hyperbola $xy = 1$ at the points A, B, C and D. Given that the triangle ABC is equilateral, find the coordinates of the point D.

3. Let f be a continuously differentiable function on $[0,1]$ such that $f(0) = 0$, $f(1) = 1$. Show that

$$\int_0^1 |f'(x) - f(x)| dx \geq \frac{1}{e}.$$

4. Suppose that $n > 1$ and that $P(x_1, x_2, \ldots x_n)$ is a polynomial in n variables of degree at most n. Assume that for every positive M the set of n-tuples $(x_1, \ldots, x_n)$ of integer numbers for which

$$|P(x_1, x_2, \ldots x_n)| \leq M$$

is finite. Show that P is not divisible by any linear form $L(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.

5. Let $f(x)$ be a positive continuous function, periodic with the period 1. Show that for any α

$$\int_0^1 \frac{f(x)}{f(x+\alpha)} dx \geq 1.$$

6. Let $f(x)$ be a continuous function such that for every 4-term arithmetic progression a, b, c, d

$$|f(d) - f(a)| \geq \pi |f(c) - f(b)|.$$

Show that f is constant.