Practice Sheet E

1. Find all solutions to $\sqrt[3]{(x-1)} + \sqrt[3]{(x-2)} + \sqrt[3]{(x+3)} = 0$.

2. Is there a rational number such that by cancelling some digits of its decimal form, the digits of π are left?

3. Solve the simultaneous equations $\{x\} = \{x^2\} = \{x^3\}$ (where $\{y\}$ denotes the fractional part of the number y, obtained by subtracting from y the greatest integer not greater than y).

4. Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2} + (k+2)\sqrt{k}}$.

5. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers $c_n r, c_n r^2 + c_{n-1}r, c_n r^3 + c_{n-1}r^2 + c_{n-2}r, \ldots, c_n r^n + c_{n-1}r_{n-1} + \cdots + c_1 r$ are integers.

6. Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called convex if for any two points in S the line segment connecting them is contained in S.)

7. Let n be a positive integer. Find the number of pairs (P, Q) of polynomials with real coefficients such that $(P(x))^2 + (Q(x))^2 = x^{2n} + 1$ and $\deg(P) > \deg(Q)$.