Practice Sheet D

1. Let n be a positive integer. What are the possible values of
$$\gcd(n^2+1,(n+1)^2+1)?$$

2. Show that if three points are inside a closed unit square, then two of them lie within $\sqrt{6}-\sqrt{2}$ units of each other.

3. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a function such that $f(g(x)) = g(f(x))$ holds for every polynomial g with real coefficients. Determine the nature of f.

4. For a fixed positive integer n let $x_1,\ldots,x_n$ be real numbers satisfying $0\le x_i\le 1$. Determine the maximum possible value of
$$\sum_{1\le i<j\le n} |x_i - x_j|.$$

5. Assume that $|f(x)|\le 1$ and $|f''(x)|\le 1$ for all x on some interval of length 2. Show that $|f'(x)|\le 2$ on the interval.

6. Find polynomials f, g, h, if they exist, such that for all x
$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1; \\ 3x+2 & \text{if } -1\le x\le 0; \\ -2x+2 & \text{if } 0 < x. \end{cases}$$

7. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where Q is a quadratic polynomial. Show that if P has at least 2 distinct complex roots, then P has n distinct complex roots.