Practice Sheet B

1. Suppose that $a_1, a_2, \dots, a_n$ are distinct real numbers. A certain polynomial p leaves a remainder of a_i when divided by $x - a_i$, for each i. What remainder is left when p is divided by $(x-a_1)(x-a_2)\cdots(x-a_n)$?

2. Let $\alpha < \beta$ be real numbers. What is the probability that, if two points are selected from a segment of length β at random, then the segment that they determine has length at least α?

3. Let $r_1, r_2, \dots$ be a sequence of positive real numbers. Suppose that

$$\lim_{n\to\infty} \left(\frac{r_1 + \cdots + r_n}{n} \right) = K$$

and

$$\lim_{n\to\infty} \left(\frac{r_1^{-1} + \cdots + r_n^{-1}}{n} \right) = L.$$

Show that $KL \geq 1$.

4. Can you load two dice (not necessarily in the same way) so that all outcomes 2, 3, ..., 12 are equally likely? (Here the sides of the dice are numbered 1–6. A die is "loaded" if it has been tampered with so that some faces turn up more often than they normally would. In this problem, you should assume only that, for each die, $p_1 + p_2 + \cdots + p_6 = 1$ and $p_i \geq 0$, where p_i is the probability of rolling i.)

5. Let $p(x)$ be a real polynomial that is nonnegative for all real x. Prove that $p(x)$ can be written a sum of squares of real polynomials:

$$p(x) = q_1(x)^2 + \cdots + q_k(x)^2.$$

6. Show that the improper integral

$$\int_0^\infty \sin(x)\sin(x^2)\, dx$$

converges.

7. Do there exist polynomials p, q, r and s such that $1 + xy + x^2y^2 = p(x)q(y) + r(x)s(y)$?