Practice Sheet A

1. Let n be a positive integer. Prove that $x^n - \frac{1}{x^n}$ is expressible as a polynomial in $x - \frac{1}{x}$ with real coefficients iff n is odd.

2. A function $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ from the positive integers to itself is *strictly increasing* if $m < n$ implies $f(m) < f(n)$ and is *multiplicative* if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Show that if f is strictly increasing, multiplicative, and satisfies $f(2) = 2$, then f satisfies $f(n) = n$ for all n.

3. Given a strictly increasing sequence $(a_n)_{n=1}^{\infty}$ of positive integers, set b_n equal to the least common multiple of $\{a_1, \ldots, a_n\}$. Show that $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges.

4. Find all complex-valued functions f of a complex variable z such that $f(z) + zf(1-z) = 1 + z$ holds for all z.

5. Show that the unit disk in the plane cannot be partitioned into two congruent disjoint subsets.

6. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive real numbers for which $\sum_{j=0}^{\infty} x_j = A$?

7. Find all twice differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$(f(x) + f(y))(f(x) - f(y)) = f(x+y)f(x-y)$$

holds for all real x and y.