

AXIOMATIZABLE CONGRUENCE PREVARIETIES

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ABSTRACT. If \mathcal{V} is a variety of algebras, let $\mathcal{L}(\mathcal{V})$ denote the prevariety of all lattices embeddable in congruence lattices of algebras in \mathcal{V} . We give some criteria for the first-order axiomatizability or nonaxiomatizability of such prevarieties. One corollary to our main results is a nonconstructive proof that every congruence n -permutable variety satisfies a nontrivial congruence identity.

1. INTRODUCTION

For a variety \mathcal{V} , let $\mathcal{L}(\mathcal{V})$ denote the class of lattices embeddable in congruence lattices of algebras in \mathcal{V} . It is evident that $\mathcal{L}(\mathcal{V})$ is closed under the formation of isomorphic lattices and sublattices. The map

$$\prod_{i \in I} \mathbf{Con}(\mathbf{A}_i) \rightarrow \mathbf{Con}\left(\prod_{i \in I} \mathbf{A}_i\right): (\gamma_i)_{i \in I} \mapsto \Gamma,$$

where $\mathbf{a} \Gamma \mathbf{b}$ if $a_i \gamma_i b_i$ for all i and $a_j = b_j$ for all but finitely many j , is an embedding, so $\mathcal{L}(\mathcal{V})$ is closed under the formation of products. This makes $\mathcal{L}(\mathcal{V})$ a prevariety, which we call the **congruence prevariety** of \mathcal{V} . In this note we discuss the first-order axiomatizability of $\mathcal{L}(\mathcal{V})$. Theorem 1.1 describes some conditions sufficient to guarantee the first-order axiomatizability of $\mathcal{L}(\mathcal{V})$, and Theorem 1.4 describes a condition necessary for axiomatizability. The combination of these two theorems yields an unexpected new proof that every congruence n -permutable variety satisfies a nontrivial congruence identity.

We use the symbol $+$ for lattice join and juxtaposition or \cdot for lattice meet. Meet takes precedence over join in expressions that are not fully parenthesized.

Theorem 1.1. *If \mathcal{V} satisfies any one of the following conditions, then $\mathcal{L}(\mathcal{V})$ is axiomatizable.*

- (1) \mathcal{V} is congruence distributive.
- (2) \mathcal{V} is congruence n -permutable for some n .
- (3) \mathcal{V} contains a nontrivial finite strongly solvable algebra.

Proof. For item (1), any prevariety of distributive lattices is a variety, hence is axiomatizable.

For item (3), if \mathcal{V} contains a nontrivial finite strongly solvable algebra, then \mathcal{V} contains a locally finite, locally solvable, minimal subvariety \mathcal{M} . According to the

results of [7] or [13], \mathcal{M} is term equivalent to the variety of sets or pointed sets. In either case, the congruence lattices of members of \mathcal{M} are exactly the partition lattices. Since every lattice is embeddable in a partition lattice, $\mathcal{L}(\mathcal{M})$ (and therefore also $\mathcal{L}(\mathcal{V})$) is the variety of all lattices.

Item (2) is proved in both [1] and [4]. We include a proof here, too. To show that the prevariety $\mathcal{L}(\mathcal{V})$ is axiomatizable, it suffices to show that it is closed under ultraproducts. That this is so is a consequence of the following claim.

Claim 1.2. *If \mathbf{A}_i , $i \in I$, are similar algebras with n -permuting congruences and \mathcal{U} is an ultrafilter on I , then the ultraproduct $\prod_U \mathbf{Con}(\mathbf{A}_i)$ is embeddable in $\mathbf{Con}(\prod_U \mathbf{A}_i)$.*

Let L be the common language of the \mathbf{A}_i 's. Expand L to a language L^+ containing extra predicate symbols, as follows. For each sequence $\Theta := (\theta_i)_{i \in I} \in \prod_I \mathbf{Con}(\mathbf{A}_i)$ introduce a binary predicate symbol $\Theta(x, y)$. Interpret $\Theta(x, y)$ in \mathbf{A}_i so that $\Theta^{\mathbf{A}_i}(a, b)$ is true iff $(a, b) \in \theta_i$. Each \mathbf{A}_i is an L^+ -structure, so the ultraproduct $\mathbf{A} := \prod_U \mathbf{A}_i$ is also an L^+ -structure. The fact that $\Theta^{\mathbf{A}_i}(x, y)$ defines a congruence on \mathbf{A}_i is first-order expressible, so $\Theta^{\mathbf{A}}(x, y)$ defines a congruence on \mathbf{A} . Consider the assignment $\prod_U \mathbf{Con}(\mathbf{A}_i) \rightarrow \mathbf{Con}(\mathbf{A})$ defined by

$$(1.1) \quad (\theta_i)_{i \in I} / \mathcal{U} (= \Theta / \mathcal{U}) \mapsto \text{the congruence defined by } \Theta^{\mathbf{A}}(x, y).$$

This is a well defined mapping, since if $\Theta = (\theta_i)_{i \in I}$ equals $\Psi = (\psi_i)_{i \in I}$ almost everywhere modulo \mathcal{U} , then \mathbf{A}_i satisfies the sentence $\forall x, y (\Theta(x, y) \leftrightarrow \Psi(x, y))$ for almost all i , so \mathbf{A} also satisfies this sentence. In this situation $\Theta^{\mathbf{A}}(x, y)$ and $\Psi^{\mathbf{A}}(x, y)$ define the same relation on \mathbf{A} . If $(\Theta \cdot \Psi)(x, y)$ is the predicate associated to the lattice meet $(\theta_i)_{i \in I} \cdot (\psi_i)_{i \in I} = (\theta_i \cdot \psi_i)_{i \in I}$, then

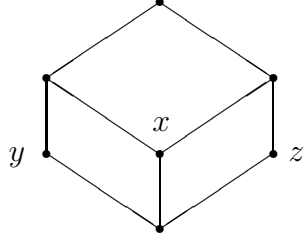
$$\mathbf{A}_i \models (\Theta \cdot \Psi)(x, y) \leftrightarrow \Theta(x, y) \ \& \ \Psi(x, y)$$

for every $i \in I$. Therefore $(\Theta \cdot \Psi)^{\mathbf{A}}(a, b)$ holds iff $\Theta^{\mathbf{A}}(a, b)$ and $\Psi^{\mathbf{A}}(a, b)$ both hold, proving that the assignment (1.1) preserves the lattice meet. If all \mathbf{A}_i have n -permuting congruences, then $(\theta_i)_{i \in I} + (\psi_i)_{i \in I} = (\theta_i + \psi_i)_{i \in I} =: \Theta + \Psi$, and

$$(1.2) \quad \mathbf{A}_i \models (\Theta + \Psi)(x, y) \leftrightarrow \exists z_0, \dots, z_n (x = z_0 \ \& \ y = z_n \ \& \ \Theta(z_i, z_{i+1}), \ i \text{ even, and } \Psi(z_i, z_{i+1}), \ i \text{ odd})$$

for all i . Thus \mathbf{A} satisfies the formula in (1.2). It follows that the congruence defined by $(\Theta + \Psi)^{\mathbf{A}}(x, y)$ is the n -fold composition (hence the join) of the congruences defined by $\Theta^{\mathbf{A}}(x, y)$ and $\Psi^{\mathbf{A}}(x, y)$. This fact implies that (1.1) preserves the lattice join, completing the proof of the claim and the theorem. \square

To express our necessary condition for axiomatizability we need some notation. First, we define a sequence of lattice words. Let $\beta_0(x, y, z) = y$, $\gamma_0(x, y, z) = z$, $\beta_{k+1}(x, y, z) = y + x \cdot \gamma_k(x, y, z)$, and $\gamma_{k+1}(x, y, z) = z + x \cdot \beta_k(x, y, z)$. Second, the lattice depicted next will be called \mathbf{D}_1 .


 FIGURE 1: THE LATTICE \mathbf{D}_1

We will need a presentation of \mathbf{D}_1 . The following lemma is easy to prove, and in any case can be derived from Lemma 5.27 of [6] (which is precisely the dual of the Lemma 1.3).

Lemma 1.3. *A presentation of \mathbf{D}_1 relative to the variety of all lattices is $\langle G \mid R \rangle$ where $G = \{x, y, z\}$ and R consists of the relations:*

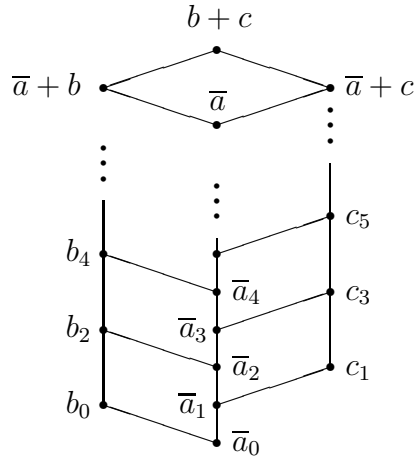
- (I) $x \leq y + z$,
- (II) $z(x + y) \leq y$,
- (III) $y(x + z) \leq z$, and
- (IV) $(x + y)(x + z) \leq x$.

Moreover any lattice generated by G and satisfying the relations in R and also satisfying $x \not\leq z$ is isomorphic to \mathbf{D}_1 .

Theorem 1.4. *If $\mathcal{L}(\mathcal{V})$ is axiomatizable, then either*

- (1) $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$, or
- (2) $\mathcal{L}(\mathcal{V}) \models \beta_m(\bar{x}, y, z) \approx \beta_{m+1}(\bar{x}, y, z)$ for some m , where $\bar{x} := (x + y)(x + z)(y + z)$.

Proof. Let \mathbf{F} be free over $\{a, b, c\}$ in $\mathcal{L}(\mathcal{V})$. Let $\bar{a} = (a + b)(a + c)(b + c)$, $b_k = \beta_k(\bar{a}, b, c)$, $c_k = \gamma_k(\bar{a}, b, c)$, $\bar{a}_k = \bar{a} \cdot b_k$ if k is even and $\bar{a}_k = \bar{a} \cdot c_k$ if k is odd. Let \mathbf{L} be the sublattice of \mathbf{F} that is generated by $\{\bar{a}, b, c\}$. The elements \bar{a}, \bar{a}_k, b_k and c_k all belong to \mathbf{L} , and some of them are ordered as follows.

FIGURE 2: SOME ELEMENTS OF \mathbf{L}

From the way the elements \bar{a} , b_k , and c_k are defined in terms of a , b and c , and the fact that $\bar{a} \leq b + c$, it is easy to see that \mathbf{L} satisfies the following relations.

- (i) $b = b_0 \leq b_2 \leq b_4 \leq \dots \leq \bar{a} + b \leq b + c$,
- (ii) $c = c_0 \leq c_1 \leq c_3 \leq \dots \leq \bar{a} + c \leq b + c$,
- (iii) $\bar{a}_0 \leq \bar{a}_1 \leq \bar{a}_2 \leq \dots \leq \bar{a}$,
- (iv) $b + \bar{a}_{2k+1} = b + \bar{a}_{2k+2} = b_{2k+2}$,
- (v) $c + \bar{a}_{2k} = c + \bar{a}_{2k+1} = c_{2k+1}$,
- (vi) $b_{2k} \cdot \bar{a} = \bar{a}_{2k}$, and
- (vii) $c_{2k+1} \cdot \bar{a} = \bar{a}_{2k+1}$.

Less obvious is the fact that

- (viii) $\bar{a} = (\bar{a} + b)(\bar{a} + c)$.

To see that this is so, observe that $\bar{a} \leq \bar{a} + b \leq (a+b)(b+c)$ and $\bar{a} \leq \bar{a} + c \leq (a+c)(b+c)$ in \mathbf{F} , so meeting corresponding elements in these inequalities yields

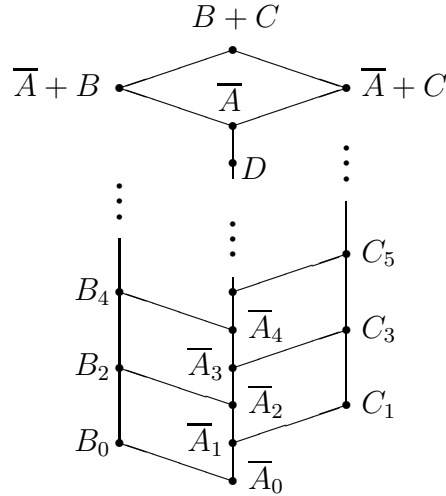
$$\bar{a} = \bar{a} \cdot \bar{a} \leq (\bar{a} + b)(\bar{a} + c) \leq (a + b)(a + c)(b + c) = \bar{a}.$$

This shows that (viii) holds.

Using these relations it can be seen that the order among the elements is as is depicted in Figure 2, and also that if any two of the elements that appear in the figure are equal in \mathbf{L} , then $b_k = b_{k+1}$ for all sufficiently large k . If this happens, then since $\mathbf{L} \leq \mathbf{F}$ and \mathbf{F} is freely generated by $\{a, b, c\}$ we get that $\mathcal{L}(\mathcal{V})$ satisfies $\beta_k(\bar{x}, y, z) \approx \beta_{k+1}(\bar{x}, y, z)$ for any sufficiently large k . In this situation item (2) of the theorem holds.

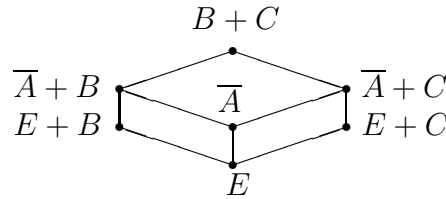
Assume henceforth that item (2) of the theorem does not hold, so all elements depicted in Figure 2 are distinct elements of \mathbf{L} . Let \mathcal{U} be a nonprincipal ultrafilter

on ω , and let \mathbf{L}^* be the sublattice of the ultrapower $\prod_{\mathcal{U}} \mathbf{L}$ that is generated by the diagonal elements $\bar{A} := (\bar{a}, \bar{a}, \bar{a}, \dots)/\mathcal{U}$, $B := (b, b, b, \dots)/\mathcal{U}$, $C := (c, c, c, \dots)/\mathcal{U}$, and the nondiagonal element $D := (\bar{a}_0, \bar{a}_1, \bar{a}_2, \dots)/\mathcal{U}$. As before, define $B_k = \beta_k(\bar{A}, B, C)$, $C_k = \gamma_k(\bar{A}, B, C)$, $\bar{A}_k = \bar{A} \cdot B_k$ if k is even and $\bar{A}_k = \bar{A} \cdot C_k$ if k is odd. The fact that each coordinate of $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \dots)$ is strictly less than the corresponding coordinate of $(\bar{a}, \bar{a}, \bar{a}, \dots)$ implies that $D < \bar{A}$ in \mathbf{L}^* . The fact that all but finitely many of the coordinates of the diagonal tuple $(\bar{a}_k, \bar{a}_k, \bar{a}_k, \dots)$ are strictly less than the corresponding coordinate of $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \dots)$ implies that $\bar{A}_k < D$ in \mathbf{L}^* . Thus, \mathbf{L}^* looks very much like \mathbf{L} with an extra element D inserted in the spot indicated in Figure 3.


 FIGURE 3: SOME ELEMENTS OF \mathbf{L}^*

If $\mathcal{L}(\mathcal{V})$ is axiomatizable, then \mathbf{L}^* belongs to $\mathcal{L}(\mathcal{V})$, which means that \mathbf{L}^* is embeddable in $\mathbf{Con}(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{V}$. Fix such an embedding and label the image with the same labels as those used in Figure 3. Let $E \in \mathbf{Con}(\mathbf{A})$ denote the join of the elements \bar{A}_k , $k < \omega$. Observe that $E \leq D < A$. The proof of the theorem may be completed by proving the following claim.

Claim 1.5. *The elements $\{\bar{A}, \bar{A} + B, \bar{A} + C, B + C, B + E, C + E, E\}$ constitute a sublattice of $\mathbf{Con}(\mathbf{A})$ that is isomorphic to \mathbf{D}_1 :*



Hence $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$.

This claim will be proved by applying Lemma 1.3 to the congruences $x := \bar{A}$, $y := B + E$, and $z := C + E$. Using the fact that $\bar{A} \geq E$, the statements (I)–(IV) from Lemma 1.3 that must be established may be written as:

- (I) $\bar{A} \leq (E + B) + (E + C)$,
- (II) $(C + E)(\bar{A} + B) \leq (B + E)$,
- (III) $(B + E)(\bar{A} + C) \leq (C + E)$, and
- (IV) $(\bar{A} + B)(\bar{A} + C) \leq \bar{A}$.

Item (I) is true since $B + C \geq \bar{A}$. Item (IV) is true because $\bar{a} = (\bar{a} + b)(\bar{a} + c)$ in \mathbf{L} . Items (II) and (III) are dual, so we prove only (II). For this we have

$$\begin{aligned}
(C + E)(\bar{A} + B) &= [(C + E)(C + \bar{A})](\bar{A} + B) \\
&= (C + E)[(C + \bar{A})(\bar{A} + B)] \\
&= (C + E)\bar{A} = \bar{A}(C + \sum_{k \text{ even}} \bar{A}_k) \\
&= \bar{A}(\sum_{k \text{ even}} (C + \bar{A}_k)) \\
&= \bar{A}(\sum_{k \text{ odd}} C_k) \\
&= \sum_{k \text{ odd}} \bar{A}C_k \quad (\text{by the upper continuity of } \mathbf{Con}(\mathbf{A})) \\
&= \sum_{k \text{ odd}} \bar{A}_k = E \leq B + E.
\end{aligned}$$

To show that the sublattice generated by x, y and z is isomorphic to \mathbf{D}_1 we must show that $x \not\leq y$, i.e. $\bar{A} \not\leq C + E$. If instead $\bar{A} \leq C + E$, then from the middle lines of the previous calculation we would have $\bar{A} = \bar{A} \cdot \bar{A} \leq \bar{A}(C + E) = E$, contradicting our earlier conclusion that $E \leq D < A$. \square

Corollary 1.6. *The congruence prevariety of the variety of semilattices is not axiomatizable.*

Proof. The variety of semilattices is congruence meet semidistributive and \mathbf{D}_1 is not meet semidistributive, so Theorem 1.4 (1) does not hold. On the other hand, the variety of semilattices satisfies no nontrivial congruence identity according to [2], so Theorem 1.4 (2) does not hold, either. \square

This corollary can be substantially strengthened. It is proved in Chapter 4 of [6] that the following conditions are equivalent for any variety \mathcal{V} :

- (1) $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$.
- (2) \mathcal{V} satisfies no nontrivial idempotent Maltsev condition.
- (3) $\mathcal{L}(\mathcal{V})$ contains every finitely presented lattice satisfying Whitman's condition.¹

Using this fact we get

¹Whitman's condition is the sentence

$$\forall w, x, y, z ((x \cdot y \leq w + z) \leftrightarrow (x \leq w + z) \text{ or } (y \leq w + z) \text{ or } (x \cdot y \leq w) \text{ or } (x \cdot y \leq z)).$$

It is satisfied by \mathbf{D}_1 .

Corollary 1.7. *If \mathcal{V} satisfies a nontrivial idempotent Maltsev condition and $\mathcal{L}(\mathcal{V})$ is axiomatizable, then $\mathcal{L}(\mathcal{V}) \models \beta_m(\bar{x}, y, z) \approx \beta_{m+1}(\bar{x}, y, z)$ for some m , where $\bar{x} := (x + y)(x + z)(y + z)$.*

From this we derive

Theorem 1.8. (cf. [5, 6, 9, 10, 11, 12]) *Any congruence n -permutable variety satisfies a nontrivial congruence identity.*

Proof. Any congruence n -permutable variety \mathcal{V} satisfies a nontrivial idempotent Maltsev condition, e.g. the one in [3]. By Theorem 1.1 (2), $\mathcal{L}(\mathcal{V})$ is axiomatizable. It follows from Corollary 1.7 that \mathcal{V} satisfies a congruence identity of the form $\beta_m(\bar{x}, y, z) \approx \beta_{m+1}(\bar{x}, y, z)$ for some m , where $\bar{x} := (x + y)(x + z)(y + z)$. \square

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