FINITELY GENERATED VARIETIES WITH SMALL $\langle R, S\rangle$-IRREDUCIBLE SETS

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ABSTRACT. Let $\mathscr{V}$ be a finitely generated variety of algebras. We show that there is a finite bound on the size of the $\langle R, S\rangle$-irreducible sets of finite algebras in $\mathscr{V}$ iff $\mathscr{V}$ is congruence 3-permutable and has a near unanimity term operation of some arity.

1. INTRODUCTION

Tame congruence theory, [8], is a localization theory for polynomial expansions of finite algebras. The theory exploits the structure and distribution of the $\langle \alpha, \beta\rangle$-minimal sets for tame congruence intervals $[\alpha, \beta]$. These sets are the same as the $\langle \alpha, \beta\rangle$-irreducible sets for arbitrary congruence intervals.

The concepts of tame congruence theory make sense for arbitrary compatible relations in place of congruences. We use letters $\alpha, \beta, \ldots$ for congruences and $R, S, \ldots$ for compatible relations.

In this paper we address a question raised in [10]: which locally finite varieties have a finite bound on the size of $\langle R, S\rangle$-irreducible sets of finite algebras? The locally finite varieties that have a finite bound on the size of $\langle \alpha, \beta\rangle$-irreducible sets, where $[\alpha, \beta]$ is a congruence interval of a finite algebra in the variety, are exactly the congruence distributive varieties, but not all locally finite congruence distributive varieties have a finite bound on the size of $\langle R, S\rangle$-irreducible sets. It can be derived from the results in [4] that a variety $\mathscr{V}$ generated by a 2-element algebra has a finite bound on the size of its $\langle R, S\rangle$-irreducible sets iff $\mathrm{typ}\{\mathscr{V}\} = \{\mathbf{3}\}$ and $\mathscr{V}$ has a near unanimity term. In this paper we show more generally that a finitely generated variety $\mathscr{V}$ has a finite bound on the size of its $\langle R, S\rangle$-irreducible sets iff $\mathscr{V}$ is congruence 3-permutable and has a near unanimity term.

In the course of our proof we characterize some other properties concerning the size and distribution of $\langle R, S\rangle$-irreducible sets in a locally finite variety, namely

(1) $\mathscr{V}$ is arithmetical iff $\mathscr{V}$ has $\langle R, S\rangle$-irreducible sets of size ≤ 2.

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(2) $\mathscr{V}$ is congruence distributive and congruence 3-permutable iff whenever $\sigma\colon \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism between finite algebras in $\mathscr{V}$, then for any $\langle R, S\rangle$-irreducible set U of $\mathbf{A}$ there is an irreducible set V of $\mathbf{B}$ such that $\sigma|_V\colon V \to U$ is a bijection.

2. Preliminaries

The goal of this paper is to characterize which finitely generated varieties $\mathscr{V}$ have the property that there is a finite number k such that any $\langle R, S\rangle$-irreducible set of a finite algebra $\mathbf{A} \in \mathscr{V}$ has size at most k. In this section we will introduce all of the terminology necessary to make sense of this statement. Furthermore, we shall reformulate our goal in terms of "covers" rather than "irreducible sets". This material is taken from [10] and [14]. The details can also be found in [2].

Definition 2.1. Let $\mathbf{A}$ be an algebra. A *neighborhood* of $\mathbf{A}$ is a subset $U \subseteq A$ such that there exists a unary term operation $e \in \mathrm{Clo}(\mathbf{A})$ that is *idempotent* (i.e., $\mathbf{A} \models e(e(x)) = e(x)$) such that $e(A) = U$.

We allow neighborhoods to have size 1.

Theorem 2.2. *Let $\mathbf{A}$ be an algebra and let $f, e_i \in \mathrm{Clo}(\mathbf{A})$, $i \in I$, be idempotent unary term operations. Let $V = f(A)$ and $U_i = e_i(A)$ be the neighborhoods defined by these idempotents, and let $\mathsf{U} = \{U_i \mid i \in I\}$. The following are equivalent.*

(1) *For compatible relations R and S of any arity we have*

$$\Big(\bigwedge_{i\in I} R|_{U_i} = S|_{U_i}\Big) \Rightarrow R|_V = S|_V.$$

(1)′ *For compatible relations R and S of arity $|A|$ we have*

$$\Big(\bigwedge_{i\in I} R|_{U_i} = S|_{U_i}\Big) \Rightarrow R|_V = S|_V.$$

(2) *There exist term operations $\lambda \in \mathrm{Clo}(\mathbf{A})$ and $\rho_i \in \mathrm{Clo}_1(\mathbf{A})$ such that*

$$\mathbf{A} \models \lambda(e_{i_1}\rho_1(x), \ldots, e_{i_q}\rho_q(x)) = f(x)$$

with $i_j \in I$.

Proof. $[(1) \Rightarrow (1)']$ This is a tautology.

$[(1)' \Rightarrow (2)]$ Let $S = \{\big(t(a)\big)_{a\in A} \mid t \in \mathrm{Clo}_1(\mathbf{A})\}$ be the $|A|$-ary relation consisting of the graphs of unary term operations. The tuples in S that can be restricted to V are the graphs of unary term operations with range in V, namely the graphs of term operations of the form $f\rho$ for some unary term operation ρ. Similarly, the tuples in S that can be restricted to U_i are the graphs of term operations of the form $e_i\rho_i$.

Let R be the relation generated by the tuples in S that are restrictable to some U_i. Since $R \subseteq S$ and R contains all tuples in S that are restrictable to some U_i, it follows that $R|_{U_i} = S|_{U_i}$ for all i. Since we are assuming $(1)'$ we have $R|_V = S|_V$. The graph of f is a tuple in S that is restrictable to V, hence is a tuple which belongs to $S|_V$, which equals $R|_V$. But the term operations whose graphs are in $R|_V$ have the form $\lambda(e_{i_1}\rho_1, \ldots, e_{i_q}\rho_q)$ for some q, some $\lambda \in \mathrm{Clo}_q(\mathbf{A})$, some $\rho_i \in \mathrm{Clo}_1(\mathbf{A})$ and some e_{i_j}, $i_j \in I$. Thus

$$\mathbf{A} \models \lambda(e_{i_1}\rho_1(x), \ldots, e_{i_q}\rho_q(x)) = f(x)$$

for appropriate term operations.

[(2) $\Rightarrow$ (1)] Assume that

$$\mathbf{A} \models \lambda(e_{i_1}\rho_1(x), \ldots, e_{i_q}\rho_q(x)) = f(x)$$

for appropriate term operations. Suppose that R and S are compatible relations of $\mathbf{A}$, that $\big(\bigwedge_{i\in I} R|_{U_i} = S|_{U_i}\big)$, and that $\mathbf{r} \in R|_V$. Then $\rho_j(\mathbf{r}) \in R$ for each j, so $e_{i_j}\rho_j(\mathbf{r}) \in R|_{U_{i_j}} = S|_{U_{i_j}} \subseteq S$ for each j, so

$$S \ni \lambda(e_{i_1}\rho_1(\mathbf{r}), \ldots, e_{i_q}\rho_q(\mathbf{r})) = f(\mathbf{r}) = \mathbf{r}.$$

Since $\mathbf{r} \in R|_V$ was arbitrary, we get $R|_V \subseteq S$. A similar argument yields $S|_V \subseteq R$, so $R|_V = S|_V$. □

This theorem inspires the following definition.

Definition 2.3. Let $\mathbf{A}$ be an algebra, let $\mathsf{U} = \{U_i \mid i \in I\}$ be a set of neighborhoods of $\mathbf{A}$, and let V be a single neighborhood of $\mathbf{A}$. U *covers* V if

$$\Big(\bigwedge_{i\in I} R|_{U_i} = S|_{U_i}\Big) \Rightarrow R|_V = S|_V$$

whenever R and S are compatible relations of $\mathbf{A}$.

The following definitions are inspired by tame congruence theory.

Definition 2.4. Let $U = e(A)$ be a neighborhood of $\mathbf{A}$. The algebra *induced* on U by $\mathbf{A}$ is

$$\mathbf{A}|_U := \langle U; \{et \mid t \in \mathrm{Clo}(\mathbf{A})\}\rangle.$$

Definition 2.5. Let U and V be neighborhoods of $\mathbf{A}$. A *morphism* from U to V is a function $t\colon U \to V$ that is the restriction to U of a term operation. An *isomorphism* is an invertible morphism.

Write $t\colon U \simeq V$ to denote that t is an isomorphism from U to V, and $U \simeq V$ to denote that U is isomorphic to V.

Definition 2.6. Let $\mathbf{A}$ be an algebra, and let $R \neq S$ be compatible relations of $\mathbf{A}$. An *$\langle R, S\rangle$-irreducible set* is a subset $U \subseteq A$ that is minimal under inclusion among neighborhoods for which $R|_U \neq S|_U$. A neighborhood is *irreducible* if it is $\langle R, S\rangle$-irreducible for some pair $\langle R, S\rangle$.

To make a technical observation, it is possible for a 1-element set to be an irreducible neighborhood. This happens exactly when it has the form $U = \{u\} \subseteq A$ and

(1) u is the image of a constant unary term operation of the algebra $\mathbf{A}$, and
(2) $\mathbf{A}$ has no zeroary term operations.

When these conditions are met, U is $\langle \emptyset, A\rangle$-irreducible.

We can use the notion of a cover to give an intrinsic characterization of irreducibility of neighborhoods.

Theorem 2.7. *Let V be a neighborhood of $\mathbf{A}$. The following are equivalent.*

(1) *V is irreducible.*
(2) *The set of proper subneighborhoods of V fails to cover V.*
(3) *V is a member of every cover of the induced algebra $\mathbf{A}|_V$.*

Proof. Item (3) is a slight reformulation of item (2), so we prove only that (1) $\Leftrightarrow$ (2).

[(1) $\Rightarrow$ (2)] If V is $\langle R, S\rangle$-irreducible, then V is minimal under inclusion among neighborhoods for $R|_V \neq S|_V$. If $\mathsf{U} = \{U_i \mid i \in I\}$ is the set of all proper subneighborhoods of V, then

$$\Big(\bigwedge_{i\in I} R|_{U_i} = S|_{U_i}\Big) \quad \& \quad R|_V \neq S|_V,$$

so V is not covered by the set of all its proper subneighborhoods.

[(2) $\Rightarrow$ (1)] If the set U of proper subneighborhoods of V is not a cover of V, then it follows from the definition of cover that there is a pair $\langle R, S\rangle$ of compatible relations such that $R|_V \neq S|_V$ while $R|_{U_i} = S|_{U_i}$ for all $U_i \in \mathsf{U}$. Hence V is $\langle R, S\rangle$-irreducible. $\square$

Next we move from covers of neighborhoods to covers of sets of neighborhoods.

Definition 2.8. Let $\mathbf{A}$ be an algebra and let U and V be sets of neighborhoods of $\mathbf{A}$.

(1) U is *isomorphic* to V, written $\mathsf{U} \simeq \mathsf{V}$, provided there is a bijection $\beta\colon \mathsf{U} \to \mathsf{V}$ such that $U \simeq \beta(U)$ for every $U \in \mathsf{U}$.
(2) U *covers* V if U covers each neighborhood in V.
(3) U is an *irredundant cover* of V if U is a cover of V and the deletion of any member of U results in a set that is no longer a cover of V.
(4) (a) U *refines* V (or *is a refinement* of V), written $\mathsf{U} \ll \mathsf{V}$ or $\mathsf{V} \gg \mathsf{U}$, provided U covers V and for every $U \in \mathsf{U}$ there exists a $V \in \mathsf{V}$ such that $U \subseteq V$.
  (b) U is a *proper refinement* of V if $\mathsf{U} \ll \mathsf{V}$ and $\mathsf{V} \not\ll \mathsf{U}$.
  (c) V is *nonrefinable* if $\mathsf{U} \ll \mathsf{V}$ implies $\mathsf{V} \subseteq \mathsf{U}$. (This is stronger than the property that V has no proper refinement.)

For the next result we introduce a quasiorder on neighborhoods of finite algebras: $U \prec V$ if U is isomorphic to a subneighborhood of V. It is easy to see that $\prec$ is reflexive and transitive, and that the induced equivalence relation is the isomorphism relation. (This last conclusion depends on the assumption that the algebra is finite.) A neighborhood V is a *maximal* irreducible it is maximal in this quasiorder among the irreducible neighborhoods. Equivalently, V is a maximal irreducible if it is irreducible and no isomorphic copy is contained in a strictly larger irreducible neighborhood.

Theorem 2.9. *Let $\mathbf{A}$ be a finite algebra.*

1. *A has a cover consisting of irreducible neighborhoods.*
2. *Every cover of $\mathbf{A}$ can be refined to a cover that is irredundant and nonrefinable.*
3. *$\mathbf{A}$ has exactly one irreducible, nonrefinable cover up to isomorphism. Such a cover consists of exactly one isomorphic copy of each maximal irreducible neighborhood of $\mathbf{A}$.*

Proof. Item (1) follows from (3), since the neighborhoods in the cover described in (3) are irreducible, so we prove the second two items only.

To prove (2), suppose that U is a cover of $\mathbf{A}$. Consider the following two operations on covers:

(i) (Deleting a redundant neighborhood) Replace U by $\mathsf{U} - \{U\}$ for some $U \in \mathsf{U}$, provided $\mathsf{U} - \{U\}$ still covers $\mathbf{A}$.
(ii) (Proper refinement) Replace U by $(\mathsf{U} - \{U\}) \cup \mathsf{V}$ where $U \in \mathsf{U}$ and V is the set of proper subneighborhoods of U, provided $(\mathsf{U} - \{U\}) \cup \mathsf{V}$ is still a cover of $\mathbf{A}$.

Starting with an arbitrary cover U_0 of $\mathbf{A}$ and applying operations of types (i) and (ii) alternately we obtain a refinement sequence $\mathsf{U}_0 \gg \mathsf{U}_1 \gg \cdots$ of covers of $\mathbf{A}$, which must terminate, since $\mathbf{A}$ is finite. The final cover V in this sequence is a refinement of U_0 that is irredundant (by (i)), hence consists of neighborhoods incomparable under inclusion. None of the neighborhoods in V can be omitted in any further refinement (by (ii)), so $\mathsf{W} \ll \mathsf{V}$ implies $\mathsf{V} \subseteq \mathsf{W}$. This shows that V is irredundant and nonrefinable.

The proof of (3) depends on the next claim.

Claim 2.10. *If U is a cover of an irreducible neighborhood V, then $V \prec U$ for some $U \in \mathsf{U}$.*

If V is $\langle R, S\rangle$-irreducible and $(R \cap S)|_V \neq S|_V$, then V is $\langle R \cap S, S\rangle$-irreducible. If $S|_V = (R \cap S)|_V \neq R|_V$, then V is $\langle R \cap S, R\rangle$-irreducible. Thus, by replacing one of R or S by their intersection we may (and do) assume that V is $\langle R, S\rangle$-irreducible where $R \subsetneq S$. Choose a relation $S' \subseteq S$ that is minimal for $S'|_V \nsubseteq R$ and let $R' = R \cap S'$. Then $R'|_V \neq S'|_V$, but for any subneighborhood $V' \subseteq V$ we have $R'|_{V'} = (R \cap S')|_{V'} = R|_{V'} \cap S'|_{V'} = S|_{V'} \cap S'|_{V'} = S'|_{V'}$. This proves that V is also $\langle R', S'\rangle$-irreducible. The minimality of S' guarantees that S' is generated by any tuple in $S' - R'$. Since $R'|_V \neq S'|_V$ we get that there is a tuple $\mathbf{s} \in S'|_V - R'|_V$, and, as just observed, $S' = \langle \mathbf{s}\rangle$.

The facts that U covers V and $R'|_V \neq S'|_V$ jointly imply that $R'|_U \neq S|_U$ for some $U \in \mathsf{U}$. Hence there is is a tuple $\mathbf{s}' \in S'|_U - R'|_U$, and by the observation of the previous paragraph we have $S' = \langle \mathbf{s}' \rangle$. Since $\mathbf{s}$ and $\mathbf{s}'$ both generate the relation S', there are unary term operations g and h such that $g(\mathbf{s}) = \mathbf{s}'$ and $h(\mathbf{s}') = \mathbf{s}$. These unary term operations will be used to produce an isomorphism from V to a subneighborhood of U

Choose idempotent unary term operations e and f such that $e(A) = U$ and $f(A) = V$. The term operation $fheg$ maps A to V and fixes $\mathbf{s}$, so some iterate $E = (fheg)^k$ is an idempotent term operation defining a subneighborhood $E(A) = V'' \subseteq V$ containing $\mathbf{s}$. This means that $R'|_{V''} \neq S'|_{V''}$, which contradicts the $\langle R', S' \rangle$-irreducibility of V unless $V'' = V$. It follows that E is an idempotent term operation with image V and hence that $eg(V) \subseteq U$ is a subset isomorphic to V via $V \stackrel{eg}{\to} eg(V) \stackrel{(fheg)^{k-1}fh}{\longrightarrow} V$. Any set isomorphic to a neighborhood is another neighborhood, so U contains a subneighborhood isomorphic to V. This proves the claim.

We use the claim to prove (3). We know by (2) that $\mathbf{A}$ has an irredundant, nonrefinable cover U. Suppose that V is a maximal irreducible neighborhood of $\mathbf{A}$. From the claim we know that some neighborhood $U \in \mathsf{U}$ contains a subneighborhood isomorphic to V. But the nonrefinability of U implies that all members of U are irreducible. By the maximality of V it must be that U itself is isomorphic to V. Two different neighborhoods in U cannot both be isomorphic to V, since U is irredundant, so U contains exactly one neighborhood isomorphic to V. This shows that U contains exactly one copy of each isomorphism type of maximal irreducible neighborhood.

We argue now that any set of neighborhoods containing one copy of each isomorphism type of maximal irreducible neighborhood is a cover of $\mathbf{A}$, hence (by irredundance) U contains no neighborhoods other than single copies of the maximal irreducibles. So, let W be a collection of neighborhoods containing a copy of each maximal irreducible. If $R \neq S$ are compatible relations, then there is an $\langle R, S \rangle$-irreducible set N. There is a maximal irreducible set $W \in \mathsf{W}$ such that $N \prec W$, and for such a set W $R|_N \neq S|_N$ implies $R|_W \neq S|_W$. This proves that whenever $R \neq S$ there is a $W \in \mathsf{W}$ such that $R|_W \neq S|_W$. Hence

$$\Big(\bigwedge_{W \in \mathsf{W}} R|_W = S|_W \Big) \Rightarrow R|_A = S|_A,$$

which establishes that W covers $\mathbf{A}$. □

Corollary 2.11. *Let $\mathbf{A}$ be a finite algebra. The following are equivalent statements about a positive integer k.*

(1) *Every irreducible neighborhood of $\mathbf{A}$ has size at most k.*
(2) *$\mathbf{A}$ is covered by its neighborhoods of size at most k.*

Proof. [(1) $\Rightarrow$ (2)] From Theorem 2.9 (1) it follows that any positive integer that works as a bound in (1) also works as a bound in (2).

[(2) $\Rightarrow$ (1)] Theorem 2.9 (2) and (3) together imply that if U is an irreducible neighborhood of $\mathbf{A}$ of maximum size, then some isomorphic copy of U must appear in every cover of $\mathbf{A}$. Thus any positive integer that works as a bound in (2) also works as a bound in (1). $\square$

We are interested only in polynomial expansions of finite algebras in this paper, and we will denote by $\mathbf{A}_A$ the expansion of $\mathbf{A}$ by constants. Hence we will be interested in neighborhoods of the expansion $\mathbf{A}_A$, and in covers consisting of such neighborhoods. In the case of polynomial expansions, Theorem 2.2 proves that U is a cover of V iff $\mathbf{A}$ satisfies a polynomial identity of the form $\lambda(e_{i_1}\rho_1(x), \ldots, e_{i_q}\rho_q(x)) = f(x)$. In reference to conditions (1) and (1)$'$ of Theorem 2.2 it is worth pointing out that the compatible relations of $\mathbf{A}_A$ are precisely the compatible reflexive relations of $\mathbf{A}$.

The notion of "induced algebra" defined here agrees exactly with corresponding the notion given in [8] when applied to polynomial expansions of finite algebras. Our notion of "$\langle R, S\rangle$-irreducible set" coincides exactly with the notion of "$\langle \alpha, \beta\rangle$-minimal set" given in [8] provided (i) one is dealing with polynomial expansions of finite algebras and (ii) $\langle R, S\rangle = \langle \alpha, \beta\rangle$ is a tame congruence quotient. The truth of this last assertion is not obvious, since our definition of irreducible set is formulated quite differently than the Hobby-McKenzie definition of minimal set, but it is not very hard to prove the equality of the concepts using the theory developed in [8].

The project of this paper is to identify which finitely generated varieties $\mathscr{V}$ have the property that there is some finite k such that the irreducible neighborhoods of $\mathbf{A}_A$ have size at most k for every finite $\mathbf{A} \in \mathscr{V}$. If we did not deal with polynomially defined neighborhoods (i.e., neighborhoods of $\mathbf{A}_A$), but rather dealt with term defined neighborhoods (i.e., neighborhoods of $\mathbf{A}$), this project would be trivial. (If V is an irreducible neighborhood of $\mathbf{A}$, then V^r is an irreducible neighborhood of $\mathbf{A}^r$ for every r, so a finite cardinality bound on such sets would imply that every irreducible neighborhood of $\mathbf{A}$ has size 1. This would force $|A| = 1$ for every $\mathbf{A} \in \mathscr{V}$.)[1]

Definition 2.12. Let k be a positive integer. A finite algebra $\mathbf{A}$ is *k-bounded* if $\mathbf{A}_A$ is covered by its neighborhoods of size at most k. A locally finite variety $\mathscr{V}$ is *k-bounded* if each finite $\mathbf{A} \in \mathscr{V}$ is k-bounded.

Our task for the rest of the paper is to characterize finitely generated k-bounded varieties.

[1]Incidentally, the argument given here has implications even in the case where we are considering polynomially defined neighborhoods, provided $\mathscr{V}$ is locally nilpotent. When $\mathscr{V}$ is such, then it is shown in [9] that idempotent twin polynomials have isomorphic images. Thus, since any idempotent polynomial of $\mathbf{A}^r$ is a twin of an idempotent of $\mathbf{A}$ acting diagonally, it follows that if V is an irreducible neighborhood of $\mathbf{A}_A$, then V^r is an irreducible neighborhood of $(\mathbf{A}_A)^r$ for every r. This already hints at our finding that k-bounded varieties must be highly nonabelian.

We end this preliminary section with some remarks on notation. $\mathbf{Con}(\mathbf{A})$ denotes the congruence lattice of $\mathbf{A}$ and $\mathrm{Con}(\mathbf{A})$ denotes the set of congruences of $\mathbf{A}$. The join and meet of congruences α and β is denoted $\alpha+\beta$ and $\alpha\beta$ respectively. We may write $a \overset{\alpha}{\text{---}} b$ to indicate that $(a,b)\in\alpha$ for $\alpha\in\mathrm{Con}(\mathbf{A})$.

3. k-bounded varieties, part 1

In this section we will derive some conditions that must be satisfied by any locally finite k-bounded variety.

Lemma 3.1. *If $\mathbf{A}$ is finite and $\mathbf{A}_A$ is k-bounded, then $\mathbf{A}$ has k-permuting congruences.*

Proof. Assume instead that $\mathbf{A}$ has congruences $\alpha,\beta\in\mathrm{Con}(\mathbf{A})$ that fail to k-permute. Let $R=\alpha\circ_k\beta=\alpha\circ\beta\circ\cdots$ (k factors), and let $S=\beta\circ_k\alpha$. Both R and S are reflexive relations on $\mathbf{A}$, hence are compatible relations of $\mathbf{A}_A$. We have $R=\alpha\circ_k\beta\neq\beta\circ_k\alpha=S$.

Since $\mathbf{A}_A$ is k-bounded, it is covered by a set U of neighborhoods of size at most k. Since R and S are distinct compatible relations of $\mathbf{A}_A$, there is some $U\in\mathsf{U}$ such that $R|_U\neq S|_U$. But $R|_U=\alpha|_U\circ_k\beta|_U$ and $S|_U=\beta|_U\circ_k\alpha|_U$, so $\alpha|_U$ and $\beta|_U$ are equivalence relations on U that fail to k-permute. This is impossible, since any two equivalence relations on a set of size $\leq k$ will k-permute. $\square$

Corollary 3.2. *A k-bounded locally finite variety is congruence k-permutable.*

Proof. By Lemma 3.1, all finite algebras in $\mathscr{V}$ have k-permuting congruences, and therefore all finitely generated free algebras have k-permuting congruences. It follows from the main result of [7] that all algebras in $\mathscr{V}$ have k-permuting congruences. $\square$

Lemma 3.3. *If $\mathbf{A}$ is finite and $\mathbf{A}_A$ is k-bounded, then the congruence lattice $\mathbf{Con}(\mathbf{A})$ belongs to the prevariety $\mathsf{S}\,\mathsf{P}(\Pi_k)$ generated by the lattice Π_k of partitions of a k-element set.*

Proof. Let U be a cover of $\mathbf{A}_A$ consisting of neighborhoods of size at most k. For each $U\in\mathsf{U}$ the restriction map $\theta\mapsto\theta|_U$ is a lattice homomorphism from $\mathbf{Con}(\mathbf{A})$ to the lattice of all equivalence relations on U by Lemma 2.3 of [8]. Since $|U|\leq k$, the lattice of equivalence relations on U may be embedded in Π_k. Thus, for each $U\in\mathsf{U}$ we get a lattice homomorphism from $\mathbf{Con}(\mathbf{A})$ to Π_k; moreover, the collection of these homomorphisms separates the points of $\mathbf{Con}(\mathbf{A})$, since U is a cover of $\mathbf{A}_A$ and the relations we are restricting (the congruences of $\mathbf{A}$) are compatible relations of $\mathbf{A}_A$. Altogether this yields $\mathbf{Con}(\mathbf{A})\in\mathsf{S}\,\mathsf{P}(\Pi_k)$. $\square$

Corollary 3.4. *A k-bounded locally finite variety is congruence distributive.*

Proof. We prove the result through a sequence of claims.

Claim 3.5. *The congruence variety of $\mathscr{V}$ is a finitely generated variety of lattices.*

It follows from Lemma 3.3 that the congruence lattice of any finite algebra in $\mathscr{V}$ belongs to the prevariety $\mathsf{S}\,\mathsf{P}(\Pi_k)$. It is a general fact that the identities satisfied by the congruence lattices of the algebras in a variety are the same as those satisfied by the congruence lattices of the finitely generated members of the variety, hence the congruence variety of $\mathscr{V}$ is contained in the variety $\mathsf{H}\,\mathsf{S}\,\mathsf{P}(\Pi_k)$. The claim follows from the fact that a subvariety of a finitely generated variety of lattices is finitely generated.

Claim 3.6. *$\mathscr{V}$ is congruence modular.*

It is proved in [5] that any nonmodular congruence variety contains Polin's congruence variety, hence is not finitely generated. (The finitely generated free algebras in Polin's variety have congruence lattices that are splitting lattices, hence are subdirectly irreducible lattices. This infinite collection of subdirectly irreducible lattices prevents the congruence variety from being finitely generated.) Thus, $\mathscr{V}$ must be congruence modular.

Claim 3.7. *$\mathscr{V}$ is congruence distributive.*

According to [1] any finitely generated variety of lattices is finitely axiomatizable, but according to [6] the only finitely axiomatizable modular congruence varieties are distributive. Thus $\mathscr{V}$ is congruence distributive. □

To prove the next consequence of k-boundedness we need more definitions. We will be considering the case where $\mathbf{A}$ is finite, $\mathbf{B}$ is a subalgebra of $\mathbf{A}^n$, and S is an n-ary relation on $\mathbf{B}$. A typical element of S will be written as an $n \times n$-matrix with entries in A and columns in B:

$$M = \begin{bmatrix} m_{11} & & m_{1n} \\ \vdots & \cdots & \vdots \\ m_{n1} & & m_{nn} \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & & | \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} | \\ \mathbf{b}_j \\ | \end{bmatrix} = \begin{bmatrix} m_{1j} \\ \vdots \\ m_{nj} \end{bmatrix}.$$

By a *(row) transversal* for M we mean a column vector built by selecting one element from each row of M. More precisely, $\begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ is a transversal for M if t_i is an element of A taken from from the ith row of M. (The simplest type of transversal of M is a column of M.) The *diagonal transversal* of M is the transversal $\begin{bmatrix} m_{11} \\ \vdots \\ m_{nn} \end{bmatrix}$ of diagonal elements of M. All other transversals of M are called *nondiagonal transversals.*

A matrix $M \in B^n$ satisfies the *diagonal transversal restriction* if the diagonal transversal of M lies in B. A relation $S \subseteq B^n$ satisfies *diagonal transversal restriction* if all matrices in S satisfy it. A matrix $M \in B^n$ satisfies the *nondiagonal transversal restrictions* if all nondiagonal transversals of M lie in B. A relation $S \subseteq B^n$ satisfies the *nondiagonal transversal restrictions* if all matrices in S satisfy those restrictions.

A relation $S \subseteq B^n$ is *defined* by all nondiagonal transversal restrictions if it contains exactly those matrices M that satisfy the nondiagonal transversal restrictions. Similarly, $S \subseteq B^n$ is *defined* by all transversal restrictions if it contains exactly those matrices M that satisfy all transversal restrictions. It is easy to see that if $\mathbf{B}$ is a subalgebra of $\mathbf{A}^n$, then n-ary relations on B defined by transversal restrictions are compatible.

Lemma 3.8. *Assume that* $\mathbf{A}$ *is finite,* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}^n$*, and* $\mathbf{B}_B$ *is* k*-bounded for some* $k < n$*. The* n*-ary relation on* $\mathbf{B}$ defined *by all nondiagonal transversal restrictions also satisfies the diagonal transversal restriction.*

Proof. Assume instead that $S \subseteq B^n$ is defined by all nondiagonal transversal restrictions, but that S does not satisfy the diagonal transversal restriction. This implies that there is a matrix $M \in S$ such that the diagonal of M is not in B, but all nondiagonal transversals of M lie in B.

Let $R \subseteq B^n$ be the relation defined by *all* transversal restrictions. Clearly $R \subseteq S$ and $M \in S - R$, hence $R \neq S$. Let's show that both R and S are reflexive n-ary relations on B. If $\mathbf{b} \in B$, then any transversal of the tuple

$$\begin{bmatrix} | & & | \\ \mathbf{b} & \cdots & \mathbf{b} \\ | & & | \end{bmatrix} \in B^n$$

is $\mathbf{b}$ itself, which lies in B. Hence these "diagonal tuples" of B^n satisfy all transversal restrictions. This shows that R and S are both reflexive relations of $\mathbf{B}_B$.

Our assumption is that $\mathbf{B}_B$ is k-bounded for some $k < n$. This means that the set U of neighborhoods of $\mathbf{B}$ of size $\leq k$ is a cover of $\mathbf{B}_B$. Hence there must exist $U \in \mathsf{U}$ such that $R|_U \neq S|_U$. If $P \in S|_U - R|_U$, then P has columns lying in U, P satisfies the nondiagonal transversal restrictions, but P does not satisfy the diagonal transversal restriction. We now argue that there can be no such P.

Since P has n columns, all from the set U, and $|U| \leq k < n$, it must be that some columns are duplicates of others. Say

$$P = \begin{bmatrix} p_{11} & & p_{1n} \\ \vdots & \cdots & \vdots \\ p_{n1} & & p_{nn} \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ | & & | \end{bmatrix} \quad \text{where } \mathbf{c}_i = \mathbf{c}_j \text{ for some } i \neq j.$$

In this case, the diagonal transversal of P equals a nondiagonal transversal:

$$\begin{bmatrix} p_{11} \\ \vdots \\ p_{ii} \\ \vdots \\ p_{jj} \\ \vdots \\ p_{nn} \end{bmatrix} = \begin{bmatrix} p_{11} \\ \vdots \\ p_{ij} \\ \vdots \\ p_{ji} \\ \vdots \\ p_{nn} \end{bmatrix}.$$

This cannot happen, since nondiagonal transversals of P lie in B and the diagonal transversal of P does not lie in B. This contradiction concludes the proof. □

In order to convert Lemma 3.8 into a statement about term operations we need to introduce cube terms. To this end we view functions in $\{x, y\}^n$ as characteristic functions of subsets of n. Namely, a subset $U \subseteq n$ corresponds to the characteristic function

$$\chi_U(i) = \begin{cases} x & \text{if } i \in U; \\ y & \text{if } i \notin U. \end{cases}$$

An *n-cube term* for $\mathscr{V}$ is a term t satisfying identities expressing the fact that when t is applied to the characteristic functions of nonempty subsets of n, in some order, one obtains the characteristic function of the empty set.

For example, a 2-cube term is one for which the vector identity $t(\chi_{\{0\}}, \chi_{\{0,1\}}, \chi_{\{1\}}) = \chi_\emptyset$ holds in $\mathscr{V}$, which means (writing elements of $\{x, y\}^2$ as columns):

$$t\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix}$$

holds in $\mathscr{V}$. Row-wise, this says that $\mathscr{V}$ satisfies the identities $t(x, x, y) = y$ and $t(y, x, x) = y$. Hence a 2-cube term is just a Maltsev term.

Similarly, a 3-cube term is a 7-ary term for which

$$t(\chi_{\{0\}}, \chi_{\{1\}}, \chi_{\{2\}}, \chi_{\{0,1\}}, \chi_{\{0,2\}}, \chi_{\{1,2\}}, \chi_{\{0,1,2\}}) = \chi_\emptyset$$

holds in $\mathscr{V}$. This vector identity is equivalent to the three row identities

$$\begin{aligned} t(x, y, y, x, x, y, x) &= y \\ t(y, x, y, x, y, x, x) &= y \\ t(y, y, x, y, x, x, x) &= y. \end{aligned}$$

Cube terms were first introduced in [3].

Corollary 3.9. *A k-bounded locally finite variety has a $(k + 1)$-cube term.*

Proof. We apply Lemma 3.8 to a specific situation. Let $n = k+1$, let $\mathbf{A} = \mathbf{F}_{\mathscr{V}}(x,y)$ be the (finite) $\mathscr{V}$-free algebra generated by the set $\{x,y\}$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^n$ generated by the set $\{x,y\}^n - \{y\}^n$ of characteristic functions of nonempty subsets of n. All the hypotheses of Lemma 3.8 are met, so the n-ary relation S of $\mathbf{B}$ defined by all nondiagonal transversal restrictions also satisfies the diagonal transversal restriction.

But the matrix

$$Z = \begin{bmatrix} y & x & x & \cdots & x \\ x & y & x & & x \\ x & x & y & & x \\ \vdots & & & \ddots & \vdots \\ x & x & x & \cdots & y \end{bmatrix}$$

satisfies all nondiagonal transversal restrictions (since the nondiagonal transversals are exactly the generators of $\mathbf{B}$). Therefore Z satisfies the diagonal transversal restriction. This is the statement that the tuple $\{y\}^n$ (= the characteristic function of the empty set) belongs to $\mathbf{B}$. The term which when applied to the generators $\{x,y\}^n - \{y\}^n$ of $\mathbf{B}$ produces $\{y\}^n$ is an n-cube term for $\mathscr{V}$. (Recall $n = k+1$.) $\square$

Corollary 3.10. *A k-bounded locally finite variety has a $(k+1)$-ary near unanimity term.*

Proof. If $\mathscr{V}$ is k-bounded, then it is congruence distributive and has a cube term by Corollaries 3.4 and 3.9. But it is proved in each of [3, 13] and [16] that a variety has an n-ary near unanimity term iff it is congruence distributive and has an n-cube term, provided $n \geq 3$. In our case, we may assume that $\mathscr{V}$ is nontrivial, so $k \geq 2$, so $n = k+1 \geq 3$. $\square$

If $\mathbf{A} \in \mathscr{V}$ is finite, then we will say that $\mathbf{A}$ is *totally bounded (with respect to $\mathscr{V}$)* if there is an integer N depending only on $\mathbf{A}$ such that, whenever $\mathbf{B} \in \mathscr{V}$ is finite and $\sigma\colon \mathbf{B} \to \mathbf{A}$ is surjective, then $\mathbf{B}$ has a neighborhood V of size at most N such that $\sigma(V) = A$.

Lemma 3.11. *If $\mathscr{V}$ is a k-bounded locally finite variety, then every finite algebra in $\mathscr{V}$ is totally bounded.*

Proof. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras in a k-bounded variety $\mathscr{V}$ and $\sigma\colon \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism. We will argue that $\mathbf{B}$ has a neighborhood V of size at most $N = k^{k^{|A|}}$ such that $\sigma(V) = A$.

Let V be a cover of $\mathbf{B}_B$ whose members have size at most k. For each $V_i \in \mathsf{V}$ let f_i be an idempotent unary polynomial of $\mathbf{B}$ such that $f_i(A) = V_i$. Since V is a cover of $\mathbf{B}_B$, Theorem 2.2 guarantees that there are polynomials λ and ρ_i such that

$$\lambda(f_1\rho_1(x), \ldots, f_q\rho_q(x)) = x$$

holds in $\mathbf{B}$. Define an equivalence relation on the subscript set $\{1, \ldots, q\}$ by $i \sim j$ iff $\sigma(f_i\rho_i)$ and $\sigma(f_j\rho_j)$ are equal polynomials of $\mathbf{A}$. The number of $\sim$-classes is at most $k^{|A|}$, since there are at most this many functions defined on the set $A = \sigma(B)$ that have range of size at most k, and each $\sigma(f_i\rho_i)$ is such a function.

Let $\lambda'(x_1, \ldots, x_p)$ be the polynomial of $\mathbf{B}$ derived from $\lambda(x_1, \ldots, x_q)$ by identifying the variables belonging to the same $\sim$-class. Also, select one polynomial $f_u\rho_u(x)$ with subscript from each $\sim$-class. In this way one obtains polynomials λ' and $f_u\rho_u$ such that for

$$g(x) := \lambda'(f_{i_1}\rho_{i_1}(x), \ldots, f_{i_p}\rho_{i_p}(x))$$

we have $\sigma(g)(x) = x$ on $\mathbf{A}$. The subscript p can be no larger than the number of $\sim$-classes, so $p \leq k^{|A|}$. Some power $h = g^r$ of g is idempotent, so we can modify the previous displayed line by applying g^{r-1} to both sides to obtain

$$h(x) := \lambda''(f_{i_1}\rho_{i_1}(x), \ldots, f_{i_p}\rho_{i_p}(x))$$

which is an idempotent unary polynomial of $\mathbf{B}$ for which $\sigma(h)(x) = x$ on $\mathbf{A}$. Because h is idempotent, its range $V = h(B)$ is a neighborhood of $\mathbf{B}_B$. The previous displayed line implies that $\sigma(V) = \sigma(h(B)) = \sigma(h)(B) = A$. Finally, the size of V may be estimated as follows:

$$\begin{aligned} |V| = |h(B)| &= |\lambda''(f_{i_1}\rho_{i_1}(B), \ldots, f_{i_p}\rho_{i_p}(B))| \\ &\leq |f_{i_1}\rho_{i_1}(B)| \times \cdots \times |f_{i_p}\rho_{i_p}(B)| \\ &\leq k^p \leq k^{k^{|A|}}. \end{aligned}$$

This completes the proof. □

Theorem 3.12. (Summary) *If $\mathscr{V}$ is a k-bounded locally finite variety, then the following conditions hold.*

(1) *$\mathscr{V}$ is k-permutable.*
(2) *$\mathscr{V}$ has a $(k+1)$-ary near unanimity term operation.*
(3) *Finite algebras in $\mathscr{V}$ are totally bounded.*

This result is not yet in final form. To improve it we need to investigate what it means for the algebras in $\mathscr{V}$ to be totally bounded, a task for the next section.

4. Totally bounded neighborhoods, algebras and varieties

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras, $\sigma \colon \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism. If U is a neighborhood of $\mathbf{A}_A$ that is the image of the idempotent polynomial $e(x) = t^{\mathbf{A}}(x, \bar{a})$, then there is a neighborhood $V \subseteq B$ such that $\sigma(V) = U$. One sees this by choosing a tuple $\bar{b}$ from B that is a preimage of $\bar{a}$ under σ and considering the polynomial $t^{\mathbf{B}}(x, \bar{b})$. This polynomial may not be idempotent, but it has an idempotent iterate, f, and $f(B) =: V$ is a neighborhood of $\mathbf{B}_B$ that σ maps onto U.

Given a neighborhood U of $\mathbf{A}_A$, there may be many neighborhoods of $\mathbf{B}_B$ that are preimages of U under σ. We begin by establishing some basic properties of minimal preimages. Henceforth a *minimal preimage* of a neighborhood U of $\mathbf{A}_A$ under a surjective homomorphism $\sigma\colon \mathbf{B} \to \mathbf{A}$ will always be a neighborhood V of $\mathbf{B}_B$ such that $\sigma(V) = U$, and which is minimal under inclusion among neighborhoods of $\mathbf{B}$ that σ maps onto U.

Lemma 4.1. *Let $\mathscr{V}$ be a locally finite variety, let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be finite algebras in $\mathscr{V}$, and let $\mathbf{C} \overset{\tau}{\to} \mathbf{B}$ and $\mathbf{B} \overset{\sigma}{\to} \mathbf{A}$ be surjective homomorphisms. The following are true.*

(1) *If U is a neighborhood of $\mathbf{A}_A$, V is a neighborhood of $\mathbf{B}_B$ and $\sigma(V) \supseteq U$, then there is a minimal preimage V' of U contained in V.*
(2) *If V and V' are neighborhoods of $\mathbf{B}_B$ that are minimal preimages of the neighborhood U of $\mathbf{A}_A$, then V is polynomially isomorphic to V'.*
(3) *Suppose that U is a neighborhood of $\mathbf{A}_A$, V is a neighborhood of $\mathbf{B}_B$, W is a neighborhood of $\mathbf{C}_C$, and $\tau(W) = V$ and $\sigma(V) \subseteq U$. Then W is a minimal preimage of U iff both W is a minimal preimage of V and V is a minimal preimage of U.*
(4) *If U is a neighborhood of $\mathbf{A}_A$ that is covered by a set U of (sub)neighborhoods, then any minimal preimage V of U is covered by a set V of (sub)neighborhoods consisting of minimal preimages of members of U. In fact, the equation decomposing U into members of U can be lifted to an equation decomposing V into minimal preimages of members of U.*
(5) *A minimal preimage of an irreducible neighborhood of $\mathbf{A}_A$ is irreducible.*

Proof. Item (1). Suppose that U is the image of the idempotent polynomial $e(x) = t^{\mathbf{A}}(x, \bar{a}) \in \mathrm{Pol}_1(\mathbf{A})$ and that V is the image of the idempotent $f \in \mathrm{Pol}_1(\mathbf{B})$. Choosing a tuple $\bar{b}$ from B that is a preimage of $\bar{a}$ under σ and consider the polynomial $ft^{\mathbf{B}}(x, \bar{b})$. This polynomial may not be idempotent, but it has an idempotent iterate g, and $g(B) =: V'$ is a neighborhood of $\mathbf{B}_B$ contained in the image V of f that σ maps onto U.

Item (2). Let $\sigma\colon \mathbf{B} \to \mathbf{A}$ be a surjective homomorphism, and suppose that $V = f(B)$ and $V' = f'(B)$ are minimal preimages of $U = e(A)$, where $f, f' \in \mathrm{Pol}_1(\mathbf{B})$ and $e \in \mathrm{Pol}_1(\mathbf{A})$ are idempotent. Since $\sigma(f)(x) = e(x) = \sigma(f')(x)$ in $\mathbf{A}$, we have that $\sigma(ff')(x) = ee(x) = e(x)$ in $\mathbf{A}$. The polynomial $g = ff'$ has an idempotent iterate g^n whose image is a subneighborhood of $f(B) = V$ that σ maps onto U. The minimality of V implies that this image $g^n(B)$ equals V. Hence $V = g^n(B) \subseteq g(B) = ff'(B) = f(V') \subseteq V$, forcing $V = f(V')$. A similar argument shows that $V' = f'(V)$, so f and f' are polynomial bijections between V and V'. In fact, f and $f'(g^{n-1})$ are inverse polynomial bijections between V and V'.

Item (3). If $\sigma(V) = U$, then $\sigma\tau(W) = \sigma(V) = U$, showing that if V is a preimage of U and W is a preimage of V (part of the assumption of (3)), then W is a preimage of U. Conversely, if $\sigma\tau(W) = U$, then $\sigma(W) = V$ (by assumption) and $\sigma(V) =$

$\sigma\tau(W) = U$, showing that if W is a preimage of U, then W is a preimage of V and V is a preimage of U.

Now we discuss the minimality of the preimages. Suppose that W is a minimal preimage of V and that V is a minimal preimage of U. Then W is a preimage of U, by the previous paragraph. If $W' \subsetneq W$ is a proper subneighborhood, then $\tau(W') \subsetneq V$ is a proper subneighborhood, hence $\sigma\tau(W') \subsetneq U$ is a proper subneighborhood. Thus W is indeed a minimal preimage of U. Conversely, assume that W is a minimal preimage of U. Then V is a preimage of U and W is a preimage of V by the previous paragraph. If V is not a minimal preimage of U, then there is a minimal preimage $V' \subsetneq V$ of U by part (1) of this lemma. By another application of part (1) there is a neighborhood $W' \subseteq W$ that is a minimal preimage of V'. By what we established in the previous paragraph, W' is a minimal preimage of U. But so was $W \supseteq W'$, so $W' = W$. This contradicts the facts that $\tau(W) = V$, $\tau(W') = V'$ and $V \neq V'$. Thus V is indeed a minimal preimage of U. Now, to show that W is a minimal preimage of V, assume otherwise that W' ($\subsetneq W$) is a preimage of V. Then $\sigma\tau(W') = \sigma(V) = U$, so W' is a preimage of U. This contradicts the minimality of W as a preimage of U.

Item (4). The statement makes two slightly different assertions, depending on whether U consists of *sub*neighborhoods of U or just consists of neighborhoods. We give the argument only in the case where U consists of subneighborhoods of U.

Suppose that U is a neighborhood of $\mathbf{A}_A$ that is covered by a set U of subneighborhoods and that V is a neighborhood of $\mathbf{B}_B$ that is a minimal preimage of U along the map $\sigma\colon \mathbf{B} \to \mathbf{A}$. The fact that U is covered by U means that a polynomial identity of the form

$$e(x) = \lambda(e_1\rho_1(x), \ldots, e_q\rho_q(x))$$

holds in $\mathbf{A}$, where $e, e_i \in \mathrm{Pol}_1(\mathbf{A})$ are idempotent, $e(A) = U$, and $e_i(A) = U_i \in \mathsf{U}$ for all i. The fact that $U_i \subseteq U$ means that we may assume that $ee_i = e_i$.

We choose preimages of the polynomials e, e_i, λ, ρ_i in $\mathbf{B}$ in the following way. First, we choose f so that $f(B) = V$. Next, using item (1), we find a minimal preimage V_i of U_i contained in V. Then we select an arbitrary idempotent f_i such that $f_i(B) = V_i$. Finally we select preimages Λ, R_i of λ, ρ_i arbitrarily. The polynomial $g(x) = f\Lambda(f_1R_1(x), \ldots, f_qR_q(x))$ is a unary polynomial of $\mathbf{B}$ with range in V such that $\sigma(g)(x) = e(x)$ in $\mathbf{A}$. The polynomial g can be iterated to an idempotent g^n whose range is contained in V and is a preimage of U. Since V is a minimal preimage of U, $g^n(B) = V$. Adjust Λ to $\Lambda' = g^{n-1}f\Lambda$. We now have that the polynomial

$$f'(x) := g^n(x) = \Lambda'(f_1R_1(x), \ldots, f_qR_q(x))$$

is an idempotent polynomial of $\mathbf{B}$ with range V, and that the previous displayed line is a decomposition equation witnessing that V is covered by the sets $f_i(B) = V_i$, which are minimal preimages of members of U.

Item (5). The previous item established that minimal preimages of reducible neighborhoods are reducible, and now we establish that minimal preimages of irreducible neighborhoods are irreducible.

Suppose that $\sigma\colon \mathbf{B}\to\mathbf{A}$ is surjective, U is an irreducible neighborhood of $\mathbf{A}_A$ and V is a neighborhood of $\mathbf{B}_B$ that is a minimal preimage of U. If V were reducible, then $\mathbf{B}$ would satisfy a decomposition equation

$$f(x)=\lambda(f_1\rho_1(x),\ldots,f_q\rho_q(x))$$

where $f, f_i\in \mathrm{Pol}_1(\mathbf{B})$ are idempotent, $f(B)=V$ and each $f_i(B)=: V_i$ is a proper subneighborhood of V. If you apply σ to all of the polynomials and sets that appear, you get a decomposition of $\sigma(f)(A)=\sigma(V)=U$ into proper subneighborhoods $U_i:=\sigma(f_i)(A)=\sigma(V_i)$. (The fact that $U_i\subsetneq U$ for all i follows from the facts that $V_i\subsetneq V$ for all i and that V is a minimal preimage of U.) □

We call a neighborhood U of $\mathbf{A}_A$, $\mathbf{A}\in\mathscr{V}$, *totally bounded (with respect to $\mathscr{V}$)* if there is an integer N such that, whenever $\mathbf{B}\in\mathscr{V}$ is finite and $\sigma\colon\mathbf{B}\to\mathbf{A}$ is surjective, then any minimal preimage of U in $\mathbf{B}$ has size at most N. The smallest possible value for such a N is, of course, $|U|$. We call U *sharply bounded* if the total boundedness of U can be established using $N=|U|$; namely, if whenever $\mathbf{B}\in\mathscr{V}$ is finite and $\sigma\colon\mathbf{B}\to\mathbf{A}$ is surjective, then any minimal preimage of U under σ has size $|U|$ (equivalently, σ maps any minimal preimage of U bijectively onto U). We call an algebra $\mathbf{A}$ totally or sharply bounded if it is so when considered as a neighborhood of itself, and we call a variety $\mathscr{V}$ totally or sharply bounded if all neighborhoods of all of its finite algebras are. The definition given here agrees with our definition of a totally bounded algebra preceding Lemma 3.11. The next lemma will be used in the following one, where we establish some basic properties of total boundedness.

Lemma 4.2. *Let $X\subseteq B$ be a subset, let $f\colon B\to B$ be an idempotent function, and θ be an equivalence relation on B compatible with f. If $V:=f(B)$ is a transversal for $\theta|_X$, then* $\ker(f)|_X=\theta|_X$.

Proof. If $(u,v)\in\theta|_X$, then $(f(u),f(v))\in\theta|_{f(B)}=\theta|_V$. But V is a θ-transversal, so $\theta|_V$ is equality. This implies that $f(u)=f(v)$, or equivalently that $(u,v)\in\ker(f)|_X$.

Now suppose that $(u,v)\in\ker(f)|_X$, so in particular $u,v\in X$. Since V is a $\theta|_X$-transversal there exist $u',v'\in V$ with $(u,u'),(v,v')\in\theta$. Then $(f(u),f(u'))\in\theta|_V$, which is the equality relation, so $f(u)=f(u')=u'$. Using this, $f(v)=f(v')=v'$, $(u,v)\in\ker(f)|_X$, and that $V=f(B)$ is a $\theta|_X$-transversal containing u' and v' we derive that

$$u'=f(u')=f(u)=f(v)=f(v')=v'.$$

Since $u\overset{\theta}{\text{---}}u'=v'\overset{\theta}{\text{---}}v$, we get $(u,v)\in\theta|_X$. □

The following lemma establishes some of the basic properties of total boundedness.

Lemma 4.3. *Let $\mathscr{V}$ be a locally finite variety and $\mathbf{A}$ a finite algebra in $\mathscr{V}$.*

(1) *If U is a 2-element neighborhood of $\mathbf{A}_A$, and the induced algebra $\mathbf{A}_A|_U$ is polynomially equivalent to a lattice or Boolean algebra, then U is totally bounded.*
(2) *Any subneighborhood of a totally bounded neighborhood is totally bounded.*
(3) *Any quotient of a totally bounded neighborhood is totally bounded.*
(4) *If a neighborhood U of $\mathbf{A}_A$ is covered by a set U of totally bounded subneighborhoods, then U itself is totally bounded.*
(5) *A neighborhood U of $\mathbf{A}_A$ is totally bounded iff it is sharply bounded.*
(6) *If $\mathscr{V}$ is totally bounded, then for any finite $\mathbf{B} \in \mathscr{V}$ and congruences $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{B})$ it is the case that*

$$\alpha \cap (\beta \circ \gamma \circ \beta) \subseteq \alpha \cap (\gamma \circ (\alpha\beta) \circ \gamma).$$

(7) *If $\mathscr{V}$ is totally bounded, then $\mathscr{V}$ is congruence distributive and congruence 3-permutable.*

Proof. Item (1). Assume that $U = e(A) = \{0, 1\}$. The fact that U is a 2-element neighborhood implies that the congruence θ generated by the pair $(0, 1)$ is join irreducible in $\mathrm{Con}(\mathbf{A})$ (since the congruence of $\mathbf{A}_A|_U$ generated by $(0, 1)$ is join irreducible in $\mathbf{Con}(\mathbf{A}_A|_U)$). Let δ be the lower cover of θ. The polynomial e satisfies $e(\theta) \not\subseteq \delta$, so its image $e(A) = U$ contains a $\langle\delta, \theta\rangle$-minimal set. This minimal set can only be U itself. We have assumed that $\mathbf{A}_A|_U$ is a lattice or Boolean algebra, so $\mathrm{typ}(\delta, \theta) \in \{\mathbf{3}, \mathbf{4}\}$.

Now suppose that $\mathbf{B} \in \mathscr{V}$ is finite and $\sigma\colon \mathbf{B} \to \mathbf{A}$ is surjective. If $\widehat{\theta} = \sigma^{-1}(\theta)$ and $\widehat{\delta} = \sigma^{-1}(\delta)$, then $\widehat{\delta} \prec \widehat{\theta}$ and $\mathrm{typ}(\widehat{\delta}, \widehat{\theta}) = \mathrm{typ}(\delta, \theta)$. If V is a $\langle\widehat{\delta}, \widehat{\theta}\rangle$-minimal set, then $\sigma(V) = U$. The fact that $U^2 \subseteq \theta$ implies that $V^2 \subseteq \widehat{\theta}$, so V is a minimal set of type $\mathbf{3}$ or $\mathbf{4}$ consisting of a single trace. From the known structure of such sets, $|V| = 2$. This implies that, for any surjective $\sigma\colon \mathbf{B} \to \mathbf{A}$ there is a minimal preimage of U in $\mathbf{B}$ of size 2. Since all minimal preimages in a give algebra are polynomially isomorphic (Lemma 4.1 (2)) this shows that minimal preimages of U throughout $\mathscr{V}$ have size at most $N = 2$.

Item (2). Suppose that $U \subseteq U'$ are neighborhoods of $\mathbf{A}_A$ and that every minimal preimage of U' in $\mathscr{V}$ has size at most N. Given any finite $\mathbf{B} \in \mathscr{V}$ and surjective $\sigma\colon \mathbf{B} \to \mathbf{A}$ choose a minimal preimage V' of U'. We have $\sigma(V') = U' \supseteq U$, so by Lemma 4.1 (1) there is a minimal preimage V of U contained in V'. Since all minimal preimages of U in $\mathbf{B}$ are polynomially isomorphic (Lemma 4.1 (2)), any minimal preimage of U in $\mathbf{B}$ has size at most $|V'| \leq N$. $\mathbf{B}$ was arbitrary, so U is totally bounded.

Item (3). What is claimed here is that if U is a totally bounded neighborhood of $\mathbf{A}_A$ and θ is a congruence on $\mathbf{A}$, then U/θ is a totally bounded neighborhood of $\mathbf{A}_A/\theta$. To see that this is so assume that any minimal preimage of U in a finite algebra $\mathbf{B} \in \mathscr{V}$ has size at most N. We argue N also bounds the size of any minimal preimage of U/θ. Let $\sigma\colon \mathbf{B} \to \mathbf{A}/\theta$ be a surjective homomorphism and let V be a neighborhood of $\mathbf{B}_B$ that is a minimal preimage of U/θ. In the pullback diagram

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\overline{\nu}} & \mathbf{B} \\ {\scriptstyle\overline{\sigma}}\big\downarrow & & {\scriptstyle\sigma}\big\downarrow \\ \mathbf{A} & \xrightarrow{\nu} & \mathbf{A}/\theta \end{array}$$

the pullback maps $\overline{\nu}$ and $\overline{\sigma}$ are surjective. If W is a minimal preimage of U in $\mathbf{P}$, then $|W| \leq N$ by our assumption about U. Since $\nu \circ \overline{\sigma}(W) = \nu(U) = U/\theta$ we get that W is a preimage of U/θ, and so W contains a minimal preimage W' of U/θ. Now $V \subseteq B$ is a minimal preimage of U/θ in $\mathbf{B}$; if W'' is a minimal preimage of V in $\mathbf{P}$, then W'' is a minimal preimage of U/θ in $\mathbf{P}$ by Lemma 4.1 (3). In this case W'' is polynomially isomorphic to W' by Lemma 4.1 (2). Now $|V| \leq |W''| = |W'| \leq |W| \leq N$, since W'' is a preimage of V, W'' and W' are polynomially isomorphic, $W' \subseteq W$, and W is a minimal preimage of U.

Item (4). Suppose that U is covered by U, which consists of totally bounded subneighborhoods. There exist polynomials λ, e, e_i, ρ_i such that $e(A) = U$, $e_i(A) = U_i \in \mathsf{U}$ and the polynomial identity

$$e(x) = \lambda(e_1\rho_1(x), \ldots, e_q\rho_q(x))$$

holds on $\mathbf{A}$. If V is a minimal preimage of U under some $\sigma \colon \mathbf{B} \to \mathbf{A}$, then, as shown in the proof of Lemma 4.1 (4), this polynomial equation can be lifted to $\mathbf{B}$:

$$f(x) = \Lambda(f_1R_1(x), \ldots, f_qR_q(x)),$$

where $f(B) = V$ and each $V_i := f_i(B)$ is a minimal preimage of U_i. Suppose that N_i bounds the sizes of minimal preimages of U_i for each i so that, for example, $|f_i(B)| = |V_i| \leq N_i$. Then

$$|V| = |f(B)| \leq |\Lambda(f_1R_1(B), \ldots, f_qR_q(B))| \leq \prod |f_i(B)| \leq \prod N_i.$$

Thus, any minimal preimage of U has size at most $N := \prod N_i$.

Item (5). We need only prove that if U satisfies the apparently weaker notion, total boundedness, then it satisfies the stronger notion, sharp boundedness. Equivalently, we need to show that there is no U that is totally bounded but not sharply bounded. Assume instead that U is a totally bounded neighborhood of $\mathbf{A}_A$ and that some integer $N > |U|$ is the least integer that witnesses its total boundedness. Then there is a finite algebra $\mathbf{B} \in \mathscr{V}$ a surjective homomorphism $\sigma \colon \mathbf{B} \to \mathbf{A}$ and a minimal preimage V of $\mathbf{B}_B$ such that $|V| = N$. These conditions on V imply that $\sigma(V) = U$, that if V' is a proper subneighborhood of V, then $\sigma(V') \subsetneq U$, and that there exist distinct $p, q \in V$ such that $\sigma(p) = \sigma(q)$. Let $e \in \mathrm{Pol}_1(\mathbf{B})$ be an idempotent polynomial for which $e(B) = V$.

Choose an integer n so that $2^n > N$, and let $\mathbf{C}$ be the subalgebra of $\mathbf{B}^n$ whose universe consists of all n-tuples $(b_1, \ldots, b_n) \in B^n$ such that $(b_i, b_i) \in \ker(\sigma)$ for all

i and j. Let $\hat{e} \in \mathrm{Pol}_1(\mathbf{C})$ be the polynomial that is e acting coordinatewise on $\mathbf{C}$, and let $\widehat{V} = \hat{e}(C)$. For any i between 1 and n let $\pi_i \colon \mathbf{C} \to \mathbf{B}$ be the ith coordinate projection and let $\eta_i = \ker(\pi_i)$. Observe that $\pi_i \colon \mathbf{C} \to \mathbf{B}$ is surjective for each i and that $\widehat{V}$ is a preimage of V under this map, hence $\widehat{V}$ is a preimage of U under $\sigma\pi_i$.

By the choices of p, q, n, the set $\{p, q\}^n$ $(\subseteq \widehat{V})$ is larger than the bound N witnessing the total boundedness of U. It follows that $\widehat{V}$ is not a minimal preimage of U, hence $\widehat{V}$ is not a minimal preimage of V. Let $V' \subsetneq \widehat{V}$ be a minimal preimage of V under π_i (hence, by Lemma 4.1 (3) V' is a minimal preimage of U under $\sigma\pi_i$).

The fact that V' is a preimage of U under $\sigma\pi_i$ implies that V' intersects each class of $\ker(\sigma\pi_i)|_{(\sigma\pi_i)^{-1}(U)}$. The congruence $\ker(\sigma\pi_i)$ relates all pairs

$$(\mathbf{a}, \mathbf{b}) = \big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) \in C^2$$

such that $(a_i, b_i) \in \ker(\sigma)$. But by the construction of $\mathbf{C}$, $(a_i, b_i) \in \ker(\sigma)$ iff $(a_j, b_j) \in \ker(\sigma)$ for any i and j. Thus the fact that V' intersects every class of $\ker(\sigma\pi_i)|_{(\sigma\pi_i)^{-1}(U)}$ implies that V' intersects every class of $\ker(\sigma\pi_j)|_{(\sigma\pi_j)^{-1}(U)}$ for every j. Hence V' is a preimage of U under $\sigma\pi_j \colon \mathbf{C} \to \mathbf{A}$ for each j, and therefore $\pi_j(V')$ $(\subseteq \pi_j(\widehat{V}) = V)$ is a preimage of U under σ for each j. But V is a minimal preimage of U under σ. so $\pi_j(V') = V$ for each j. This yields

$$\pi_j(V') = V = \pi_j(\widehat{V})$$

for any j, so for any $p \in \widehat{V}$ there exists $p_j \in V'$ such that $\pi_j(p_j) = \pi_j(p)$. This implies that $(p_j, p) \in \eta_j = \ker(\pi_j)$. Now V' is a neighborhood of $\mathbf{C}_C$, so there is some idempotent $f \in \mathrm{Pol}_1(\mathbf{C})$ such that $f(C) = V'$. Let $q = f(p)$. We have $p \overset{\eta_j}{\text{---}} p_j$, so $q = f(p) \overset{\eta_j}{\text{---}} f(p_j) = p_j \overset{\eta_j}{\text{---}} p$, and therefore $q \overset{\eta_j}{\text{---}} p$ holds for every η_j. This implies that $(p, q) \in \bigwedge \eta_j = 0$, or that $p = q = f(p) \in V'$. Since $p \in \widehat{V}$ was arbitrary we get $\widehat{V} \subseteq V'$. This is a contradiction to the fact that $|\widehat{V}| \geq 2^n > N \geq |V'|$.

Item (6). Given $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{B})$, choose a pair $(r, s) \in \alpha \cap (\beta \circ \gamma \circ \beta)$. Let $\mathbf{A} = \mathbf{B}/\gamma$ and let $\sigma \colon \mathbf{B} \to \mathbf{B}/\gamma = \mathbf{A}$ be the natural map. Since $\mathscr{V}$ is totally bounded it is sharply bounded, according to part (4) of this lemma. Hence there is an idempotent polynomial $f \in \mathrm{Pol}_1(\mathbf{B})$ such that $V := f(B)$ is mapped bijectively onto the neighborhood A by σ. Equivalently V is a transversal for $\ker(\sigma) = \gamma$. Applying Lemma 4.2 in the case where $X = B$ and $\theta = \gamma$ we obtain that $\ker(f) = \gamma$. Since f is idempotent, this implies that $b \overset{\gamma}{\text{---}} f(b)$ for every $b \in B$.

The fact that $(r, s) \in \alpha \cap (\beta \circ \gamma \circ \beta)$ implies that $(r, s) \in \alpha$ and that there exist p and q such that

$$r \overset{\beta}{\text{---}} p \overset{\gamma}{\text{---}} q \overset{\beta}{\text{---}} s.$$

Applying f (and recalling that $\ker(f) = \gamma$) we get

$$f(r) \overset{\beta}{\text{---}} f(p) = f(q) \overset{\beta}{\text{---}} f(s),$$

hence $(f(r), f(s)) \in \beta$. We also have $(f(r), f(s)) \in \alpha$, so

$$r \stackrel{\gamma}{\text{---}} f(r) \stackrel{\alpha\beta}{\text{---}} f(s) \stackrel{\gamma}{\text{---}} s.$$

This proves that $(r, s) \in \alpha \cap (\gamma \circ (\alpha\beta) \circ \gamma)$, as required.

Item (7). It is enough to prove that the finite algebras in $\mathscr{V}$ have distributive, 3-permuting congruences, since $\mathscr{V}$ is locally finite.

If $\mathbf{B}$ is a finite algebra in $\mathscr{V}$, then applying part (6) of this lemma in the case $\alpha = 1$ we get $\beta \circ \gamma \circ \beta \subseteq \gamma \circ \beta \circ \gamma$ for an arbitrary pair of congruences $\beta, \gamma \in \mathrm{Con}(\mathbf{B})$. This establishes 3-permutability. Next, applying item (6) twice to an arbitrary triple of congruences we obtain

$$\begin{aligned} \alpha(\beta+\gamma) &= \alpha \cap (\beta \circ \gamma \circ \beta) \\ &\subseteq \alpha \cap (\gamma \circ (\alpha\beta) \circ \gamma) \\ &\subseteq \alpha \cap ((\alpha\beta) \circ (\alpha\gamma) \circ (\alpha\beta)) \\ &= \alpha \cap (\alpha\beta + \alpha\gamma) = \alpha\beta + \alpha\gamma. \end{aligned}$$

This proves that $\mathbf{Con}(\mathbf{B})$ is distributive. □

Our next goal is to prove that the condition in Lemma 4.3 (7) characterizes totally bounded locally finite varieties. For this we need a method for constructing small preimages of neighborhoods in finite algebras that are contained in congruence distributive and congruence 3-permutable varieties. Our arguments will need the terms guaranteed by the next theorem.

Theorem 4.4. ([15], Proposition 5) *A variety is both congruence distributive and congruence n-permutable iff there exist ternary terms $\ell_0, \ldots, \ell_n$ such that the following identities hold in $\mathscr{V}$:*

(1) $\ell_i(x, y, x) = x$ *for all i,*
(2) $\ell_i(x, x, z) = \ell_{i+1}(x, z, z)$ *for all i,*
(3) $\ell_0(x, y, z) = x$ *and* $\ell_n(x, y, z) = z$.

In the case $n = 2$ this theorem is due to Pixley (cf. [17]). The fact that terms satisfying the identities in (2) and (3) characterize congruence n-permutability is due to Hagemann and Mitschke (cf. [7]). We will call terms satisfying the identities in Theorem 4.4 *Lipparini terms*. We will need them only in the case $n = 3$.

Theorem 4.5. *The following conditions are equivalent for a locally finite variety $\mathscr{V}$.*

(1) *$\mathscr{V}$ is totally bounded.*
(2) *$\mathscr{V}$ is sharply bounded.*
(3) *$\mathscr{V}$ is congruence distributive and congruence 3-permutable.*

Proof. We proved in Lemma 4.3 that (1) and (2) are equivalent and that they imply (3). Here we prove that (3) implies that every finite algebra in $\mathscr{V}$ is sharply bounded. Then Lemma 4.3 implies that $\mathscr{V}$ is sharply bounded. (We refer to Lemma 4.3 because we defined the total or sharp boundedness of a variety in terms of the total or sharp boundedness of the *subneighborhoods* of its members, not in terms of the total or sharp boundedness of its members. Lemma 4.3 shows that the distinction is unimportant.)

The property that every finite algebra in $\mathscr{V}$ is sharply bounded is equivalent to the property that for any finite $\mathbf{B} \in \mathscr{V}$ and any congruence θ on $\mathbf{B}$ there is a neighborhood of $\mathbf{B}_B$ that is a transversal for θ. (One sees this by applying the definition to the natural map $\sigma\colon \mathbf{B} \to \mathbf{B}/\theta$.) Assume that this property fails for some variety $\mathscr{V}$ that is both congruence distributive and congruence 3-permutable. We shall derive a contradiction from this assumption.

So let $\mathbf{B}$ be a finite algebra in $\mathscr{V}$ with a congruence θ such that there is no neighborhood that is a transversal for θ. Let V be a neighborhood of $\mathbf{B}_B$ that is minimal under inclusion for the property that V intersects every θ-class, and let $f \in \mathrm{Pol}_1(\mathbf{B})$ be an idempotent whose image is V. Since V is not a transversal for θ there exist distinct θ-related elements of V.

If $x, \ell_1(x,y,z), \ell_2(x,y,z), z$ is a sequence of Lipparini terms for $n = 3$ for $\mathscr{V}$, then $f(x), f\ell_1^{\mathbf{B}}(x,y,z), f\ell_2^{\mathbf{B}}(x,y,z), f(z)$ is a sequence of polynomials of $\mathbf{B}$ that satisfy the Lipparini identities on V. Let $H(x,y,z) = f\ell_1^{\mathbf{B}}(x,y,z)$ and $K(x,y,z) = f\ell_2^{\mathbf{B}}(x,y,z)$. The Lipparini identities assert that on V we have $x = H(x,y,y) = H(x,y,x)$, $H(x,x,y) = K(x,y,y)$, and $K(z,y,z) = K(y,y,z) = z$.

Case 1. For some $(a,b) \in \theta|_V$ the function $H(x,a,b)$ is not a permutation of V.

In this case $H(x,a,b)$ maps V into itself, is not a permutation of V, but is the identity modulo θ; i.e., $H(x,a,b) \equiv H(x,a,a) = x \pmod{\theta}$. The polynomial $h(x) = H(x,a,b)$ can be iterated to an idempotent $h^n(x)$ whose range is a neighborhood $V' \subsetneq V$ which intersects each θ-class (since $h^n(x)$ is the identity modulo θ). This contradicts the minimality of V.

Case 2. For all $(a,b) \in \theta|_V$ the function $H(x,a,b)$ is a permutation of V.

Consider the ternary polynomial $H(x,y,z) = H_{y,z}(x)$ to be a function of x with y and z as parameters. This function can be iterated to a function $H^n_{y,z}(x)$ that is idempotent in x for every choice of $y, z \in V$. Let $L(x,y,z) = H^{n-1}_{y,z}(x)$. Then, if $a, b \in V$ are such that $H(x,a,b)$ is a permutation of V, then $H^n_{a,b}(x)$ is an idempotent iterate of this permutation, hence $H^n_{a,b}(x) = x$ on V. In this case $L(x,a,b)$ is the inverse of the permutation $H(x,a,b)$ on V.

Choose distinct $a, b \in V$ such that $(a,b) \in \theta$, and define the unary polynomial

$$p(x) = L(K(H(x,a,b), x, K(a,b,x)), H(x,a,b), K(a,b,x)).$$

Modulo θ we have

$$\begin{aligned}
p(x) &= L(K(H(x,a,\underline{b}),x,K(a,\underline{b},x)),H(x,a,\underline{b}),K(a,\underline{b},x)) \\
&\equiv_\theta L(K(H(x,a,\underline{\underline{a}}),x,K(a,\underline{\underline{a}},x)),H(x,a,\underline{\underline{a}}),K(a,\underline{\underline{a}},x)) \\
&= L(K(\underline{H(x,a,a)},x,\underline{K(a,a,x)}),\underline{H(x,a,a)},\underline{K(a,a,x)}) \\
&= L(K(\underline{\underline{x}},x,\underline{\underline{x}}),\underline{\underline{x}},\underline{\underline{x}}) \\
&= x,
\end{aligned}$$

since the Lipparini identities hold and all polynomials in questions are idempotent on V. This shows that $p(x)$ is the identity modulo θ. In the calculation above and in those below we underline the part that we intend to change and double underline the result of the change.

Now we prove that $p(a) = p(b)$.

$$\begin{aligned}
p(a) &= L(K(H(a,a,b),a,\underline{K(a,b,a)}),H(a,a,b),\underline{K(a,b,a)}) \\
&= L(K(H(a,a,b),a,\underline{\underline{a}}),H(a,a,b),\underline{\underline{a}}) \\
&= L(\underline{K(H(a,a,b),a,a)},H(a,a,b),a) \\
&= L(\underline{\underline{H(H(a,a,b),H(a,a,b),a)}},H(a,a,b),a) \\
&= H^n_{H(a,a,b),a}(H(a,a,b)) = H(a,a,b).
\end{aligned} \tag{†}$$

We move from the third line to the fourth using the Lipparini identity $H(x,x,y) = K(x,y,y)$. The last step follows from the fact that $L(H(x,y,z),y,z) = H^n_{y,z}(x) = x$ when $(y,z) \in \theta$. Next,

$$\begin{aligned}
p(b) &= L(K(\underline{H(b,a,b)},b,K(a,b,b)),\underline{H(b,a,b)},K(a,b,b)) \\
&= L(K(\underline{\underline{b}},b,K(a,b,b)),\underline{\underline{b}},K(a,b,b)) \\
&= L(\underline{K(b,b,K(a,b,b))},b,K(a,b,b)) \\
&= L(\underline{K(a,b,b)},b,K(a,b,b)) \\
&= L(\underline{\underline{K(a,b,b)}},b,K(a,b,b)) = K(a,b,b).
\end{aligned} \tag{‡}$$

The last step uses the identity $L(x,y,x) = x$, which follows from the facts that $H(x,y,x) = x$ and that L is a first variable iterate of $H(x,y,z)$. Finally we have

$$p(a) \overset{(\dagger)}{=} H(a,a,b) = K(a,b,b) \overset{(\ddagger)}{=} p(b),$$

where the middle equality follows from the Lipparini identity $H(x,x,y) = K(x,y,y)$.

We have shown that $p(x)$ is a polynomial that maps V into itself noninjectively, but which is the identity modulo θ. An idempotent iterate of p will have range that is a proper subneighborhood of V that intersects each θ-class. As in Case 1 this is a contradiction. The contradictions obtained were to our early assumption that V is a minimal for intersecting every θ-class, but there exist distinct $a, b \in V$ such that $(a,b) \in \theta$. We conclude, therefore, that any neighborhood minimal for intersecting every θ-class is a θ-transversal. $\square$

5. K-BOUNDED VARIETIES, PART 2

The first result of this section provides a list of conditions sufficient to imply k-boundedness for single algebras. It is the basis for our characterization of k-bounded locally finite varieties.

Lemma 5.1. *Let* $\mathbf{B}$ *be a finite algebra. Assume that*

(i) $\mathbf{B}_B$ *generates a congruence* 3*-permutable variety,*
(ii) $\mathbf{B}$ *has an* $(n+1)$*-ary near unanimity polynomial operation, and*
(iii) *every quotient of* $\mathbf{B}$ *that is a subdirect product of* n *subdirectly irreducible algebras is* k*-bounded.*

Then $\mathbf{B}$ *is* k*-bounded.*

Proof. let V be the set of neighborhoods of $\mathbf{B}_B$ of size $\leq k$. Let $\mathbf{B} \leq \prod_{i\in I} \mathbf{S}_i$ be a representation of $\mathbf{B}$ as a subdirect product of finite subdirectly irreducible quotient algebras. If $W \subseteq I$, let $\pi^W : \mathbf{B} \to \prod_{i\in W} \mathbf{S}_i$ be the projection onto the coordinates in W, let $\mathbf{B}^W = \mathrm{im}(\pi^W)$ and let $\eta^W = \ker(\pi^W)$. (When $W = \{i\}$ write η^i instead of $\eta^{\{i\}}$.)

Claim 5.2. *For every nonempty* $W \subseteq I$ *there is an idempotent* $e^W \in \mathrm{Pol}_1(\mathbf{B})$ *whose image* V^W *is a neighborhood of* $\mathbf{B}_B$ *that is a transversal for* η^W*, and for which there is a polynomial identity*

$$e^W(x) = \lambda^W(e_1^W \rho_1^W(x), \ldots, e_{q^W}^W \rho_{q^W}^W(x)),$$

satisfied in $\mathbf{B}$*, such that* e_i^W *is idempotent and* $e_i^W(B) \in \mathsf{V}$ *for all* i*.*

Establishing this claim will complete the proof, since when $W = I$ the claim yields a decomposition of the neighborhood $V^I = B$ into neighborhoods from V.

First consider the case $|W| \leq n$. Then $\pi^W : \mathbf{B} \to \mathbf{B}^W$ is surjective and $\pi^W(\mathsf{V}) = \{\pi^W(V) \mid V \in \mathsf{V}\}$ is a collection of neighborhoods of $\mathbf{B}^W_{B^W}$ that have size at most k. In fact, it is the collection of all such neighborhoods, since (by the sharp boundedness of V) any neighborhood of $\mathbf{B}^W_{B^W}$ is the image of a neighborhood of $\mathbf{B}_B$ of the same size, and V contains all neighborhoods of $\mathbf{B}_B$ of size at most k.

The algebra $\mathbf{B}^W$ is a subdirect product of at most n finite subdirectly factors of $\mathbf{B}$. In fact, we may assume that it is a subdirect product of exactly n factors by adding repeated factors, so by (2)(iii) the algebra $\mathbf{B}^W$ is covered by $\pi^W(\mathsf{V})$. Any decomposition equation expressing this fact can be lifted to $\mathbf{B}$ along the map $\pi^W : \mathbf{B} \to \mathbf{B}^W$. By Lemma 4.1 (4) this yields a decomposition of some minimal preimage of B^W into minimal preimages of neighborhoods of $\mathbf{B}^W_{B^W}$. The sizes of the neighborhoods will be preserved by sharp boundedness, yielding the statement of the claim.

We continue by induction on $|W|$. Suppose that $|W| > n$, $\{i_1, \ldots, i_{n+1}\} \subseteq W$, $W_j = W - \{i_j\}$ for $j = 1, \ldots, n+1$, and that the claim has been established for all

W_j in place of W. Let $M(x_1, \ldots, x_{n+1})$ be a near unanimity polynomial operation of $\mathbf{B}$. Define a polynomial

$$g^W(x) = M^{\mathbf{B}}(e^{W_1}(x), \ldots, e^{W_{n+1}}(x)).$$

Since $e^{W_i}(x)$ is the identity function modulo η^{W_i}, it is the identity function modulo η^j for each $j \in W_i$. Now an arbitrary $j \in W$ belongs to at least n of the sets $W_1, \ldots, W_{n+1}$, so the near unanimity identities for M guarantee that $g^W(x) = M^{\mathbf{B}}(e^{W_1}(x), \ldots, e^{W_{n+1}}(x))$ is the identity function modulo η^j for all $j \in W$, hence is the identity function modulo η^W. Some iterate $(g^W)^r$ is idempotent and still the identity function modulo η^W for every $j \in W$; let $N = (g^W)^r(B)$ be the neighborhood it defines. We must have $\pi^W(N) = B^W$, so there is a minimal preimage $V^W \subseteq N$ of B^W along the map π^W. This neighborhood is a transversal for η^W. If $f \in \mathrm{Pol}_1(\mathbf{B})$ is an idempotent whose range is V^W, then so is the composite $e^W := f(g^W)^r$. Moreover, the identity

$$e^W(x) = f(g^W)^r(x) = f(g^W)^{r-1}(M^{\mathbf{B}}(e^{W_1}(x), \ldots, e^{W_{n+1}}(x)))$$

can be expanded on the right so that it is a decomposition equation for V^W in terms of the neighborhoods $e_\ell^{W_j}(B) \in \mathsf{V}$. □

Corollary 5.3. *Let $\mathscr{V}$ be a locally finite variety. The following conditions are equivalent.*

(1) *$\mathscr{V}$ is k-bounded.*
(2) (i) *$\mathscr{V}$ is congruence 3-permutable*
  (ii) *$\mathscr{V}$ has a $(k+1)$-ary near unanimity term operation, and*
  (iii) *any subdirect product of k finite subdirectly irreducible algebras in $\mathscr{V}$ is k-bounded.*

Proof. The implications (1)⇒(2)(i) and (1)⇒(2)(ii) have been established in Theorems 3.12 and 4.5. The implication (1)⇒(2)(iii) is trivial. Lemma 5.1 in the case $n = k$ applied to each finite $\mathbf{A} \in \mathscr{V}$ proves that (2)⇒(1). □

Next we formulate a version of this corollary that characterizes finitely generated k-bounded varieties in ways that do not refer to the k-boundedness of a subclass.

Corollary 5.4. *Let $\mathscr{V}$ be a finitely generated variety. The following conditions are equivalent.*

(1) *$\mathscr{V}$ is k-bounded for some k.*
(2) *$\mathscr{V}$ is congruence 3-permutable and has a near unanimity term operation of some arity.*
(3) *There is some ℓ such that any irreducible neighborhood in any finite algebra in $\mathscr{V}$ has size at most ℓ.*

Proof. [(1) ⇒ (2)] This follows from Corollary 5.3.

[(2) $\Rightarrow$ (1)] If (2) holds, then $\mathscr{V}$ is a finitely generated congruence distributive variety, hence by Jónsson's Lemma there is a finite number s bounding the size of the subdirectly irreducible algebras in $\mathscr{V}$. If n is the arity of near unanimity term for $\mathscr{V}$, then every subdirect product of n subdirectly irreducible algebras in $\mathscr{V}$ is s^n-bounded. (Reason: if $\mathbf{S}$ is such a subdirect product, then the set S itself is a neighborhood of $\mathbf{S}_S$ that covers $\mathbf{S}_S$, and its size is $\leq s^n$.) By Corollary 5.3 $\mathscr{V}$ is k-bounded for $k = s^n$.

[(1) $\Rightarrow$ (3)] According to Corollary 2.11, Condition (1) for a given k is equivalent to Condition (3) with $\ell = k$. □

6. Refinements involving the number 2

We have shown that if $\mathscr{V}$ is a locally finite k-bounded variety for some k, then $\mathscr{V}$ is congruence 3-permutable and has a near unanimity term operation of some arity. We proved the converse for finitely generated varieties. In this section we prove the converse for arbitrary locally finite varieties provided $k = 2$.

A second refinement of our results concerns our characterization of locally finite varieties that are both congruence distributive and congruence 3-permutable as exactly the sharply bounded varieties. Here we will characterize the locally finite varieties that are both congruence distributive and congruence 2-permutable by a strengthened notion of sharp boundedness.

We start with a proof that a locally finite variety is 2-bounded if and only if it is congruence permutable and has a 3-ary near unanimity term. The key to the proof is the following unpublished result of K. A. Kearnes, E. W. Kiss and M. A. Valeriote.

Theorem 6.1. *If $\mathbf{A}$ is a finite algebra with a Maltsev polynomial, then the set of α, β-minimal sets for $\alpha \prec \beta$ in $\mathbf{Con}(\mathbf{A})$ form a cover of $\mathbf{A}_A$.*

Proof. It is enough to show that $\mathbf{A}_A$ decomposes into its minimal sets for congruence coverings. Assume the contrary and let $N \subsetneq A$ be a neighborhood of $\mathbf{A}_A$ that is maximal for the property that $\mathbf{A}_A|_N$ is decomposable into its minimal sets for congruence coverings. It is a basic fact of tame congruence theory (a consequence of Theorem 2.8 of [8]) that every finite algebra is the connected union of the traces of its minimal sets for congruences, hence there must exist an $\langle \alpha, \beta\rangle$-minimal set U for some $\alpha \prec \beta$ in $\mathrm{Con}(\mathbf{A})$ that has a trace $T \subseteq U$ that properly overlaps N. (I.e., $T \cap N \neq \emptyset$, but $T \not\subseteq N$.) Choose $0 \in T \cap N$. Thus U properly overlaps N and $U \cap N$ contains an element 0 from the body of U. Let $m(x, y, z)$ denote a Maltsev polynomial of $\mathbf{A}$.

Claim 6.2. $\mathbf{A}$ *has an idempotent unary polynomials e and f such that*

(1) *$e(A) = U$ and $e(N) = \{0\}$, and*
(2) *$f(A) = N$ and $f(U) = \{0\}$.*

Since U and N are neighborhoods there exist idempotent unary polynomials e_0 and f_0 such that $e_0(A) = U$ and $f_0(A) = N$. We show that these can be modified to have the extra properties in the claim.

Our first step is to construct e. As a first case, assume that $e_0 f_0 \in \mathrm{Pol}_1(\mathbf{A})|_U$ is not a permutation of U. The polynomial $g(x) = e_0 m(0, e_0 f_0(x), e_0(x))$ has the property that $g(A) \subseteq U$ and $g(N) = \{0\}$. If $u \in T - \{0\}$, then $(u, 0) \in \beta - \alpha$, so $(e_0 f_0(u), 0) \in \alpha$, and we get $(g(u), u) \in \alpha$. Since $(0, u) \in \beta - \alpha$, $g(0) = 0$, and $g(u) \equiv u \pmod{\alpha}$ it follows that $g(\beta) \not\subseteq \alpha$ and so g is not collapsing on U. Therefore an appropriate power $e = g^k$ is an idempotent unary polynomial with range U that collapses N to $\{0\}$.

Now assume that $e_0 f_0$ is a permutation of U. Then $U \simeq f_0(U)$, so $V := f_0(U) \subseteq N$ is an $\langle \alpha, \beta \rangle$-minimal set contained in N and containing 0. Let $S = f_0(T)$ be the $\langle \alpha, \beta \rangle$-trace of V containing 0. Corollary 4.8 of [12] guarantees that there is an idempotent unary polynomial e_1 of $\mathbf{A}$ such that in the quotient $\mathbf{A}/\alpha$ we have $\overline{e}_1(A/\alpha) = U/\alpha$ and $\overline{e}_1(S/\alpha) = \{0/\alpha\}$. Replacing e_1 by $e_0 e_1$ if necessary we may assume that $e_0 e_1 = e_1$, so $e_1(A) \subseteq U$. Since e_1 maps A into U and is is the identity modulo α on U, it follows from the $\langle \alpha, \beta \rangle$-minimality of U that $e_1(A) = U$. Now $e_1 f_0$ is collapsing on T, so $e_1 f_0$ is not a permutation of U. Thus we can repeat the argument of the first case using e_1 in place of e_0 to construct an idempotent unary polynomial e such that $e(A) = U$ and $e(N) = \{0\}$.

Now, given e and f_0 as above let $f(x) = f_0 m(f_0(x), m(e(x), x, f_0(x)), 0)$. One calculates that $f(A) \subseteq N$, f is the identity on N (so f is idempotent with range N), and $f(U) = \{0\}$. This completes the proof of the claim.

Now we complete the proof of the theorem. The polynomial $h(x) = m(e(x), 0, f(x))$ is the identity on $N \cup U$, so some iterate $h^\ell(x)$ is idempotent with range $M \supsetneq N$. The equation $m(e(h^{\ell-1}(x)), 0, f(h^{\ell-1}(x))) = h^\ell(x)$ is a decomposition equation for M into neighborhoods N and U, so we have contradicted the assumption that N is maximal among neighborhoods decomposable into minimal sets. □

Now we are in a position to prove the desired result.

Theorem 6.3. *Let $\mathscr{V}$ be a locally finite variety. The following are equivalent.*

(1) *$\mathscr{V}$ is 2-bounded.*
(2) *$\mathscr{V}$ has a Maltsev term and a 3-ary near unanimity term.*
(3) *$\mathscr{V}$ is* arithmetical *(= congruence permutable and congruence distributive).*

Proof. The equivalence of (2) and (3) is due to Pixley, [17]. Theorem 3.12 proves (1)⇒(2), so we need only prove that (3)⇒(1).

If $\mathscr{V}$ has a Maltsev term, then it follows from Theorem 6.1 that every finite $\mathbf{A}$ in $\mathscr{V}$ is covered by its minimal sets for congruences. But these have size 2, since $\mathscr{V}$ is congruence distributive (Theorem 8.6 of [8]). Hence $\mathscr{V}$ is 2-bounded. □

We turn next to our second refinement of earlier results. Recall that a neighborhood U of $\mathbf{A}_A$ is sharply bounded in $\mathscr{V}$ if whenever $\mathbf{B} \in \mathscr{V}$ is finite and $\sigma\colon \mathbf{B} \to \mathbf{A}$ is surjective, then there is a minimal preimage V of U in $\mathbf{B}$ such that $|V| = |U|$. In general we expect $\mathbf{B}$ to contain many minimal preimages of U. Let us say that a preimage V is *anchored* at $b \in B$ if $b \in V$. Of course, for a preimage to be anchored at $b \in B$ it is necessary to have $b \in \sigma^{-1}(U)$. If this is the only restriction on an anchor of a minimal preimage of U we will say that *minimal preimages of neighborhoods are freely anchored.* That is, minimal preimages of neighborhoods are freely anchored in $\mathscr{V}$ if whenever U is a neighborhood of $\mathbf{A}_A$, $\mathbf{B} \in \mathscr{V}$ is finite and $\sigma\colon \mathbf{B} \to \mathbf{A}$ is surjective, then for any $b \in \sigma^{-1}(U)$ there is a minimal preimage V of U anchored at b.

Theorem 6.4. *A locally finite variety is arithmetical iff it is sharply bounded and minimal preimages are freely anchored.*

Proof. [$\Rightarrow$] Let U be a neighborhood of $\mathbf{A}_A$, let $\mathbf{B} \in \mathscr{V}$ be finite and let $\sigma\colon \mathbf{B} \to \mathbf{A}$ be surjective. Let $m(x, y, z)$ be a Pixley term for $\mathscr{V}$, so that $m(x, y, y) = m(x, y, x) = m(y, y, x) = x$ throughout $\mathscr{V}$. Now let $V = e(B)$ be an arbitrary minimal preimage of U. $\mathscr{V}$ is sharply bounded, so V is a transversal for $\ker(\sigma)|_{\sigma^{-1}(U)}$.

Choose $b \in \sigma^{-1}(U)$ arbitrarily. Since V is a transversal for $\ker(\sigma)|_{\sigma^{-1}(U)}$ there is a unique element $c \in V$ such that $(b, c) \in \ker(\sigma)$.

Claim 6.5. *The polynomial $g(x) = m^{\mathbf{B}}(e(x), c, b)$ is an idempotent unary polynomial whose image V' is a minimal preimage of U anchored at b.*

Note that $V' := g(B) = m^{\mathbf{B}}(e(B), c, b) = m^{\mathbf{B}}(V, c, b)$ is an image of V under the polynomial $m^{\mathbf{B}}(x, c, b)$, hence $|V'| \leq |V| = |U|$. On the other hand, $\sigma(g)(x) = \sigma(m^{\mathbf{B}}(e(x), c, b)) = m^{\mathbf{A}}(\sigma(e)(x), \sigma(c), \sigma(b)) = \sigma(e)(x)$, so σ maps $g(B) = V'$ onto $\sigma(e)(A) = U$. Hence σ is a bijection from V' onto U. Since $\sigma(g)(x) = \sigma(e)(x)$, it follows that $\sigma|_M$ is an isomorphism from the unary algebra $\langle V'; g(x)\rangle$ onto $\langle U; \sigma(e)(x)\rangle$, which forces g to be the identity on $V' = g(B)$. We conclude that g is an idempotent whose image V' is a minimal preimage of U. To see that V' is anchored at b, observe that $b = g(c) \in g(B) = V'$. This completes the proof of the claim and also the proof of direction [$\Rightarrow$].

[$\Leftarrow$] Now assume $\mathscr{V}$ is sharply bounded and minimal preimages are freely anchored. Let $\mathbf{A} = \mathbf{F}_{\mathscr{V}}(u, v)$, $\mathbf{B} = \mathbf{F}_{\mathscr{V}}(u, v, w)$, and let $\sigma\colon \mathbf{B} \to \mathbf{A}$ be the map defined on generators by $u \mapsto u, v \mapsto v$ and $w \mapsto v$. Let $U = A$ and let $b = w$ be the chosen anchor. There must be some neighborhood $V \subseteq B$ that is a $\ker(\sigma)|_{\sigma^{-1}(N)}$-transversal containing $b = w$. Since $\ker(\sigma) = \mathrm{Cg}(v, w)$ and $\sigma^{-1}(U) = \sigma^{-1}(A) = B$, this means that V is a $\mathrm{Cg}(v, w)$-transversal in $\mathbf{B}$ that contains w.

Claim 6.6. *Let $\alpha = \mathrm{Cg}^{\mathbf{B}}(u, w)$, $\beta = \mathrm{Cg}^{\mathbf{B}}(u, v)$ and $\gamma = \mathrm{Cg}^{\mathbf{B}}(v, w)$. If e is an idempotent such that $e(B) = V$, then $u \stackrel{\alpha\gamma}{\longrightarrow} e(u) \stackrel{\alpha\beta}{\longrightarrow} w$.*

We have $u \overset{\beta}{\text{---}} v$, so $e(u) \overset{\beta}{\text{---}} e(v)$. Similarly $u \overset{\alpha}{\text{---}} w$, so $e(u) \overset{\alpha}{\text{---}} e(w)$. According to Lemma 4.2, $\ker(e) = \ker(\sigma) = \gamma = \mathrm{Cg}(v, w)$, so $e(v) = e(w) = w$. Adding this to our earlier conclusions, we have $e(u) \overset{\alpha\beta}{\text{---}} e(v) = e(w) = w$.

Since $e(u) = e(e(u))$ we have $(u, e(u)) \in \ker(e) = \ker(\sigma) = \gamma$. Now $u \overset{\alpha}{\text{---}} w \overset{\alpha\beta}{\text{---}} e(u)$, so $u \overset{\alpha}{\text{---}} e(u)$. These two conclusions imply that $(u, e(u)) \in \alpha\gamma$. With the conclusion of the previous paragraph we have $u \overset{\alpha\gamma}{\text{---}} e(u) \overset{\alpha\beta}{\text{---}} w$.

Let $m(x, y, z)$ be a ternary term such that $m^{\mathbf{B}}(u, v, w) = e(u)$. The fact that $m(u, v, w) \overset{\alpha\gamma}{\text{---}} u$ implies that the identities $m(x, y, y) = x = m(x, y, x)$ hold in $\mathscr{V}$, while the fact that $m(u, v, w) \overset{\alpha\beta}{\text{---}} w$ implies that the identities $m(y, y, x) = x = m(x, y, x)$ hold in $\mathscr{V}$. These identities imply that m is a Pixley term for $\mathscr{V}$. $\square$

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