CONGRUENCE LATTICES OF LOCALLY FINITE ALGEBRAS

KEITH A. KEARNES

Abstract. It is shown that there exist algebraic lattices that cannot be represented as the congruence lattice of a locally finite algebra.

1. Introduction

Recently, James Schmerl asked me if every finite lattice is isomorphic to the congruence lattice of a locally finite algebra. Noting that it is still unknown whether every finite lattice is isomorphic to the congruence lattice of a finite algebra, he reformulated the question as: *Are there any finite lattices which are known to be isomorphic to the congruence lattice of an infinite locally finite algebra but not yet known to be isomorphic to the congruence lattice of a finite algebra?*

I don't know the answer to either form of Schmerl's question, but have found a result of the opposite type: there exist finite lattices that are representable as the congruence lattice of a finite algebra, but not representable as the congruence lattice of an infinite locally finite algebra. Moreover, there exist algebraic lattices that are not the congruence lattice of any locally finite algebra at all.

To represent an algebraic lattice as the congruence lattice of an algebra, there is a lower bound on the cardinality of the representing algebra that must be satisfied. Namely, if $\mathbf{L} \cong \mathbf{Con}(\mathbf{A})$, then A must be large enough so that the lattice of all equivalence relations on A contains a complete 0,1-sublattice isomorphic to $\mathbf{L}$. Conversely, if A is at least this large, and is infinite, then the known representation theorems prove that $\mathbf{L} \cong \mathbf{Con}(\mathbf{A})$ for some algebra with universe A. (Although it is not relevant to this paper, the results in [1] and [6] show that lower bounds on the number of fundamental operations must also be satisfied.) In this paper, it is shown how to establish upper bounds on the cardinality of the representing algebra in some cases where the algebra is assumed to be locally finite. When the upper bound on cardinality is incompatible with the aforementioned lower bound, a nonrepresentability result is obtained.

2. Nonrepresentable Lattices

A congruence α on an algebra $\mathbf{A}$ is *abelian* if

$$s(\mathbf{a},\mathbf{c}) = s(\mathbf{a},\mathbf{d}) \Leftrightarrow s(\mathbf{b},\mathbf{c}) = s(\mathbf{b},\mathbf{d}) \tag{2.1}$$

whenever $s(\mathbf{x},\mathbf{y})$ is an $(m+n)$-ary term operation of $\mathbf{A}$, $\mathbf{a},\mathbf{b}$ are m-tuples satisfying $(a_i,b_i)\in\alpha$ for all i, and $\mathbf{c},\mathbf{d}$ are n-tuples satisfying $(c_j,d_j)\in\alpha$ for all j. This may be reformulated in the following way. Call n-ary polynomials $s_{\mathbf{a}}(\mathbf{y}) := s(\mathbf{a},\mathbf{y})$ and $s_{\mathbf{b}}(\mathbf{y}) := s(\mathbf{b},\mathbf{y})$ *α-twins* if they are obtained from the same term operation $s(\mathbf{x},\mathbf{y})$ by substituting α-related tuples $\mathbf{a}$ and $\mathbf{b}$ for $\mathbf{x}$ respectively. Now let $N_j = c_j/\alpha = d_j/\alpha$ be the α-class of c_j for each j. The bi-implication (2.1) merely asserts that the two functions

$$s_{\mathbf{a}}, s_{\mathbf{b}} \colon N_1\times\cdots\times N_n \to A$$

have the same kernel. Thus, α is abelian if any pair of α-twins have the same kernel when restricted to a product of α-classes. At first, this sounds like a strange way to define abelianness, but this definition agrees with the abelianness concept from group theory and generalizes it in a useful way. (Confer [2, 3].)

Theorem 2.1 of [4] proves that if $\mathbf{A}$ is a finite algebra, μ is a minimal abelian congruence on $\mathbf{A}$, and $\mathbf{B}$ is a maximal proper subalgebra of $\mathbf{A}$, then $\mathbf{B}$ is either a union of μ-classes or is contained in a *μ-transversal* (which is a set containing exactly one element from every μ-class). Equivalently, if a subset $X\subseteq A$ properly contains a μ-transversal, then X generates $\mathbf{A}$. Theorem 2.1 below partially extends this result to locally finite algebras. Namely, it proves that if $\mathbf{A}$ is a locally finite algebra and μ is a minimal abelian congruence of finite index, then any subset $X\subseteq A$ that properly contains a μ-transversal generates $\mathbf{A}$. Of course, if μ has finite index, then there is a finite subset $X_0\subseteq A$ that properly contains a μ-transversal. If $\mathbf{A}$ is generated by this finite subset, then by local finiteness $\mathbf{A}$ is finite. Conversely, if $\mathbf{A}$ is finite, then the desired result is just Theorem 2.1 of [4]. Thus, the extension to locally finite algebras should be worded as:

Theorem 2.1. *Let $\mathbf{A}$ be a locally finite algebra. If $\mathbf{A}$ has a minimal abelian congruence of finite index, then $\mathbf{A}$ is finite.*

Note that this result implies the nontrivial and new fact that any locally finite, abelian, simple algebra is finite. See Corollary 2.3 for a generalization. For the necessity of assuming abelianness, see Example 2.11.

Proof. Suppose that T is a μ-transversal, that X_0 is a finite subset of A properly containing T, and $\mathbf{B}$ is the subalgebra of $\mathbf{A}$ that is generated by X_0. It will be shown that $\mathbf{B}=\mathbf{A}$. Since $\mathbf{A}$ is locally finite and $\mathbf{B}$ is finitely generated, this forces $\mathbf{A}$ to be finite.

Since T is a μ-transversal, each element $a\in A$ is μ-related to a uniqely determined element $\overline{a}\in T$. Equivalently, there is a uniquely determined function $A\to T\colon a\mapsto\overline{a}$ for which $(a,\overline{a})\in\mu$. If $s(\mathbf{x},\mathbf{y})$ is a term operation of $\mathbf{A}$ and $p(\mathbf{y}) = s(\mathbf{a},\mathbf{y}) = s(a_1,\ldots,a_m,\mathbf{y})$ is a polynomial, then write $\overline{p}$ for the polynomial $s(\overline{\mathbf{a}},\mathbf{y}) := s(\overline{a}_1,\ldots,\overline{a}_m,\mathbf{y})$. Note that $\overline{p}$ is a μ-twin of p whose parameters lie in B.

Suppose that $N_1 \times \cdots \times N_n$ is a product of μ-classes of $\mathbf{A}$. Then

$$V := B^n \cap (N_1 \times \cdots \times N_n) = (B \cap N_1) \times \cdots \times (B \cap N_m) \tag{2.2}$$

is a product of $\mu|_{\mathbf{B}}$-classes of $\mathbf{B}$. Let $\mathcal{P}$ be the collection of subsets of A of the form $p(V)$ where V is as in (2.2), $p(\mathbf{y})$ is an n-ary polynomial of $\mathbf{A}$, and n is arbitrary. Order $\mathcal{P}$ by inclusion.

Suppose that $W = p(V) \in \mathcal{P}$ where V is of the form described in (2.2) and $p(\mathbf{y}) = s(\mathbf{a}, \mathbf{y})$ for some term operation $s(\mathbf{x}, \mathbf{y})$. The polynomial $\overline{p}$ is a μ-twin of p whose parameters lie in B. Since μ is abelian, p and $\overline{p}$ have the same kernel when restricted to any (subset of a) product of μ-classes of $\mathbf{A}$, hence p and $\overline{p}$ have the same kernel when restricted to V. This shows that $|p(V)| = |\overline{p}(V)|$. But $V \subseteq B^n$ and the sequence of parameters of $\overline{p}$ lie in $T \subseteq B$. Thus $\overline{p}(V) \subseteq s(B^m, B^n) \subseteq B$, since s is a term operation and $\mathbf{B}$ is a subalgebra. It follows that $|W| = |p(V)| = |\overline{p}(V)| \leq |B|$. Since $W \in \mathcal{P}$ was arbitrary, no chain in $\mathcal{P}$ can be longer than $|B|$. In particular, any $W \in \mathcal{P}$ is contained in set $M \in \mathcal{P}$ that is maximal under inclusion.

The purpose of this paragraph is to show that the maximal elements of $\mathcal{P}$ are subsets of B. The idea to do this comes from [5]. Let $M \in \mathcal{P}$ be a maximal element. By the definition of $\mathcal{P}$, $M = p(V)$ for some set V of the form described in (2.2) and some polynomial $p(\mathbf{y}) = s(\mathbf{a}, \mathbf{y}) = s(a_1, \ldots, a_m, \mathbf{y})$ of $\mathbf{A}$. Since $\mathbf{B}$ properly contains a μ-transversal, there is a pair $(u, v) \in \mu|_{\mathbf{B}}$ with $u \neq v$. Let N denote the μ-class of u, and let $U = B \cap N$ denote the $\mu|_{\mathbf{B}}$-class of u. Since $(a_m, \overline{a}_m) \in \mu = \mathrm{Cg}^{\mathbf{A}}(u, v)$ there is a sequence $a_m = w_1, w_2, \ldots, w_{k+1} = \overline{a}_m$ where for each i there is a unary polynomial r_i of $\mathbf{A}$ such that $\{w_i, w_{i+1}\} = \{r_i(u), r_i(v)\}$. The sets $M_i := s(a_1, \ldots, a_{m-1}, w_i, V)$ all belong to $\mathcal{P}$ since they are polynomial images of V. The sets M_i all have the same size, since for any i and j the μ-twins $s(a_1, \ldots, a_{m-1}, w_i, \mathbf{y})$ and $s(a_1, \ldots, a_{m-1}, w_j, \mathbf{y})$ have the same kernel when restricted to V. $M_1 = s(a_1, \ldots, a_{m-1}, w_1, V) = p(V) = M$ is maximal in $\mathcal{P}$. Moreover, for each i the set $M_i \cup M_{i+1}$ is contained in a set in $\mathcal{P}$, namely the set $s(a_1, \ldots, a_{m-1}, r_i(U), V)$. That this belongs to $\mathcal{P}$ follows from the facts that $s(a_1, \ldots, a_{m-1}, r(y_0), \mathbf{y})$ is a polynomial of $\mathbf{A}$ and $U \times V$ is of the form described in (2.2). That it contains both M_i and M_{i+1} follows from the fact that $\{w_i, w_{i+1}\} = \{r_i(u), r_i(v)\} \subseteq r_i(U)$. Altogether, it has been shown that $M_1, \ldots, M_k$ are sets in $\mathcal{P}$ of the same size, that $M_1 = M$ is maximal under inclusion in $\mathcal{P}$, and that $M_i \cup M_{i+1}$ is contained in a subset of $\mathcal{P}$ for each i. By induction, $M_1 = M_2 = \cdots = M_k$. Thus, if $M = s(a_1, \ldots, a_k, V)$ is maximal in $\mathcal{P}$, then changing a_i to $\overline{a}_i$ one by one does not change M. But then $M = s(\overline{a}_1, \ldots, \overline{a}_k, V) \subseteq B$, since $\overline{a}_i \in B$ for each i, $V \subseteq B^n$, s is a term operation, and $\mathbf{B}$ is a subalgebra.

It was shown in the previous paragraph that the maximal elements of $\mathcal{P}$ are subsets of B. But then all elements of $\mathcal{P}$ are subsets of B, since $\mathcal{P}$ is ordered by inclusion. In particular, the minimal elements of $\mathcal{P}$, which are the sets of the form $\{a\}$, $a \in A$, are subsets of B. This proves that $A \subseteq B$, so $\mathbf{A} = \mathbf{B}$. $\square$

It will be necessary to apply Theorem 2.1 in the situation where μ is known to be locally solvable rather than abelian, so recall from Chapter 7 of [3] the meaning of this concept. A congruence α on an algebra $\mathbf{A}$ is *solvable* if there is a finite chain of congruences $0 = \alpha_0 \leq \cdots \leq \alpha_n = \alpha$ such that α_{i+1}/α_i is an abelian congruence of $\mathbf{A}/\alpha_i$. A congruence α on $\mathbf{A}$ is *locally solvable* if its restriction to any finitely generated subalgebra is solvable.

Lemma 2.2. *If μ is a minimal congruence on $\mathbf{A}$, then μ is abelian if and only if it is locally solvable.*

Proof. It follows easily from the definitions that any abelian congruence is locally solvable.

Arguing the contrapositive of the converse implication, suppose that μ is a minimal nonabelian congruence on $\mathbf{A}$. There exist a term $s(\mathbf{x}, \mathbf{y})$ and elements such that

$$s(\mathbf{a}, \mathbf{c}) = s(\mathbf{a}, \mathbf{d}) \qquad \text{and} \qquad u = s(\mathbf{b}, \mathbf{c}) \neq s(\mathbf{b}, \mathbf{d}) = v, \tag{2.3}$$

where $(a_i, b_i) \in \mu$ and $(c_j, d_j) \in \mu$ for all i and j. Let $\mathbf{B}$ be a finitely generated subalgebra of $\mathbf{A}$ containing the finite set

$$Y = \{a_i, b_i\}_{1\leq i\leq m} \cup \{c_j, d_j\}_{1\leq j\leq n} \cup \{u, v\},$$

and enough other elements so that any two distinct μ-related elements of Y generate the same congruence ν of $\mathbf{B}$. This is possible since μ is a minimal congruence of $\mathbf{A}$. Using the commutator defined in Chapter 3 of [3], the implication in (2.3) implies that $(u, v) \in [\nu, \nu]$. But $[\nu, \nu] \leq \nu = \mathrm{Cg}^{\mathbf{B}}(u, v)$, so $[\nu, \nu] = \nu$. Since $0_{\mathbf{B}} < \nu = [\nu, \nu] \subseteq \mu|_{\mathbf{B}}$, it follows that μ restricts to a nonsolvable congruence on $\mathbf{B}$, so μ is not a locally solvable congruence of $\mathbf{A}$. □

An algebra is *locally solvable* if its largest congruence is. Congruences α and β are *locally solvably related*, writen $\alpha \overset{s}{\sim} \beta$, if $(\alpha \vee \beta)/(\alpha \wedge \beta)$ is a locally solvable congruence on $\mathbf{A}/(\alpha \wedge \beta)$. It is shown in Theorem 7.7 of [3] that when $\mathbf{A}$ is locally finite the relation $\overset{s}{\sim}$ is a complete congruence on $\mathbf{Con}(\mathbf{A})$, and $\mathbf{Con}(\mathbf{A})/\overset{s}{\sim}$ is meet semidistributive. It follows from Exercise 7.14 (2) of [3] that when $\mathbf{A}$ is locally finite and $\delta \leq \alpha \wedge \beta$ the relation $\alpha \overset{s}{\sim} \beta$ in $\mathbf{Con}(\mathbf{A})$ is equivalent to $\alpha/\delta \overset{s}{\sim} \beta/\delta$ in $\mathbf{Con}(\mathbf{A}/\delta)$.

Corollary 2.3. *If $\mathbf{A}$ is a locally finite and locally solvable algebra, and $\mathbf{Con}(\mathbf{A})$ has a finite maximal chain, then $\mathbf{A}$ is finite.*

Proof. Suppose that $0 = \alpha_0 \prec \cdots \prec \alpha_n = 1$ is a finite maximal chain in $\mathbf{Con}(\mathbf{A})$. Since $\mathbf{A}$ is locally solvable, the relation $\overset{s}{\sim}$ is the largest congruence on $\mathbf{Con}(\mathbf{A})$. Hence $\alpha_i \overset{s}{\sim} \alpha_{i+1}$ for each i, forcing α_{i+1}/α_i to be a minimal locally solvable congruence of $\mathbf{A}/\alpha_i$. By Lemma 2.2, this congruence is abelian. It follows from Theorem 2.1 that if α_{i+1} has finite index, then α_i also has finite index. By descending the chain, the least congruence 0 is of finite index, hence $\mathbf{A}$ is finite. □

Definition 2.4. Let $\mathbf{L}$ be an algebraic lattice with largest element 1. Let Ω be the smallest complete congruence on $\mathbf{L}$ such that $\mathbf{L}/\Omega$ is meet semidistributive. The *cf-filter* of $\mathbf{L}$ is the lattice filter generated by all elements $x \in L$ such that

(i) $x \equiv 1 \pmod{\Omega}$, and
(ii) the interval $[x, 1]$ has a finite maximal chain.

An element in the cf-filter of $\mathbf{L}$ is called a *cf-element*.

"Cf" stands for "cofinite".

Theorem 2.5. *If* $\mathbf{A}$ *is a locally finite algebra, then every cf-element of* $\mathbf{Con}(\mathbf{A})$ *is a congruence of finite index.*

Proof. As noted above, $\overset{s}{\sim}$ is a complete congruence on $\mathbf{L} := \mathbf{Con}(\mathbf{A})$ for which the quotient $\mathbf{L}/\overset{s}{\sim}$ is meet semidistributive. If Ω is the smallest congruence with these properties, then $x\ \Omega\ y \Rightarrow x \overset{s}{\sim} y$. Therefore, the cf-filter is generated by elements $x \in L$ for which $x \overset{s}{\sim} 1$ and $[x, 1]$ has a finite maximal chain. For each such element, $\mathbf{A}/x$ is a locally solvable algebra whose congruence lattice has a finite maximal chain. According to Corollary 2.3, $\mathbf{A}/x$ is finite, hence x has finite index. Since the cf-filter is generated by elements of finite index, it consists of elements of finite index. □

For certain lattices $\mathbf{L}$, this theorem can be used to establish an upper bound on the size of a locally finite algebra $\mathbf{A}$ for which $\mathbf{L} \cong \mathbf{Con}(\mathbf{A})$.

Corollary 2.6. *Let* $\mathbf{A}$ *be a locally finite algebra. If the least element of* $\mathbf{L} := \mathbf{Con}(\mathbf{A})$ *is a cf-element, then* $|A| < \omega$. *If the least element of* $\mathbf{L}$ *is the meet of* $\leq \kappa$ *cf-elements of* $\mathbf{L}$*, for some infinite* κ*, then* $|A| \leq 2^\kappa$.

Proof. The first claim is a direct consequence of Theorem 2.5. For the second claim, if the least element of $\mathbf{Con}(\mathbf{A})$ is the intersection of $\leq \kappa$ cf-elements, then from Theorem 2.5 the algebra $\mathbf{A}$ is a subdirect product of $\leq \kappa$ finite algebras. If κ is infinite, this forces $|A| \leq \omega^\kappa = 2^\kappa$. □

In particular, this shows that if the least element of $\mathbf{L}$ is the meet of $\leq \kappa$ cf-elements, and $|L| > 2^{2^\kappa}$, then $\mathbf{L}$ is not representable as the congruence lattice of a locally finite algebra (since a set of size $\leq 2^\kappa$ supports $\leq 2^{2^\kappa}$ equivalence relations). More generally, if $x \in L$ is obove the meet of κ cf-elements, and the interval $[x, 1]$ has cardinality greater than 2^{2^κ}, then $\mathbf{L}$ is not representable as the congruence lattice of a locally finite algebra.

Next is a nonrepresentability result whose statement does not refer explicitly to cf-elements.

Corollary 2.7. *Let* $\mathbf{L}$ *be an algebraic lattice. Suppose that*

(i) $\mathbf{L}$ *has a finite maximal chain, and*
(ii) *there is no complete homomorphism from* $\mathbf{L}$ *onto a 2-element chain.*

If $\mathbf{L} \cong \mathbf{Con}(\mathbf{A})$ *for some locally finite algebra* $\mathbf{A}$*, then* $\mathbf{A}$ *is finite. In particular, if* $\mathbf{L}$ *is an infinite algebraic lattice satisfying (i) and (ii), then* $\mathbf{L}$ *is not isomorphic to the congruence lattice of a locally finite algebra.*

Proof. The proof of the corollary is accomplished by showing that if (i) and (ii) hold, then the least element of $\mathbf{L}$ is a cf-element. Then the conclusion follows from the first part of Corollary 2.6.

Let Ω be the least complete congruence on $\mathbf{L}$ such that $\mathbf{L}/\Omega$ is meet semidistributive. If Ω is not the total binary relation on $\mathbf{L}$, then $\mathbf{L}/\Omega$ is a nontrivial, complete, meet semidistributive lattice. Since $\mathbf{L}$ is algebraic, it is meet continuous. The natural map $\nu\colon \mathbf{L} \to \mathbf{L}/\Omega$ is complete, so $\mathbf{L}/\Omega$ is also meet continuous. Since $\mathbf{L}/\Omega$ has a finite maximal chain, it has an atom α. The map $\varphi\colon \mathbf{L}/\Omega \to [0,\alpha]\colon x \mapsto x \wedge \alpha$ is a complete lattice homomorphism onto a 2-element chain. (That φ preserves complete joins uses the meet semidistributivity and meet continuity of $\mathbf{L}/\Omega$.) But then $\varphi \circ \nu$ is a complete homomorphism of $\mathbf{L}$ onto a 2-element chain, contrary to (ii). Therefore Ω is the total binary relation on $\mathbf{L}$.

If 0 and 1 are the least and largest elements of $\mathbf{L}$, then $0 \equiv 1 \pmod{\Omega}$ by the conclusion of the previous paragraph. Since $[0,1] = L$ has a finite maximal chain, 0 is a cf-element. ☐

Example 2.8. Let X be an infinite set of cardinality κ, and let $\mathbf{L} = \mathrm{Eq}(X)$ be the lattice of all equivalence relations on X. If $\mathbf{X} = \langle X; \emptyset\rangle$, then $\mathbf{L} = \mathbf{Con}(\mathbf{X})$, so $\mathbf{L}$ is representable as the congruence lattice of a locally finite algebra of size κ. From the known representations theorems, if $\mathbf{L}$ is representable in cardinality κ, then it is representable in all larger cardinalities. However, the known representation theorems do not produce locally finite algebras. Here it will be shown that $\mathbf{L}$ has infinitely many different representations as the congruence lattice of a locally finite algebra of cardinality κ, but no representation as the congruence lattice of a locally finite algebra of any other cardinality.

The observation that $\mathbf{L}$ has infinitely many different representations in cardinality κ is based on the fact that, if $\mathbf{L} = \mathbf{Con}(\mathbf{X})$, then there is a canonical isomorphism from $\mathbf{L}$ to $\mathbf{Con}(\mathbf{X}^{[k]})$ for each k where $\mathbf{X}^{[k]}$ is the k-th matrix power of $\mathbf{X}$. The cardinality of this algebra is $|X^k| = |X| = \kappa$. Such representations are "different" for different values of k because maximal congruences on $\mathbf{X}^{[k]}$ have index 2^k, and this changes as k does.

Since the lattice of equivalence relations on a κ-element set has κ compact elements when $\kappa \geq \omega$, it is clear that $\mathbf{L}$ cannot be represented as a congruence lattice of any algebra of cardinality less than κ. In this paragraph it will be shown that $\mathbf{L}$ cannot be represented as a congruence lattice of a locally finite algebra in cardinalities greater κ. Indeed, suppose that $\varphi\colon \mathbf{L} \to \mathbf{Con}(\mathbf{A})$ is an isomorphism, where $\mathbf{A}$ is a locally finite algebra. It is easy to see that the elements $\theta \in L = \mathrm{Eq}(X)$ that are of finite index are cf-elements, and that the compact elements of L are the equivalence relations on X

with finitely many nonsingleton classes, each one finite. Therefore, $\mathbf{L}$ has κ compact elements, and if $\alpha \in L$ is compact, then α has a lattice theoretic complement that is a cf-element. Using the isomorphism φ, we obtain that $\mathbf{Con}(\mathbf{A})$ has κ-many compact elements, and each one has a lattice-theoretic complement that is a cf-element. Since $\mathbf{A}$ is locally finite, the cf-elements have finite index. If a congruence α on $\mathbf{A}$ has a cf-element θ as a complement, and the index of θ is m, then the classes of α have size $\leq m$. Therefore, if $a \in A$ is fixed and $Y := \{a/\alpha \mid \alpha \in \mathrm{Con}(\mathbf{A}) \text{ compact}\}$, then Y consists of at most κ-many finite subsets of A, so $|\bigcup Y| \leq \kappa$. But if $b \in A$, then $b \in a/\operatorname{Cg}(a,b)$ and $a/\operatorname{Cg}(a,b) \in Y$. Therefore $A = \bigcup Y$ has size at most κ.

Example 2.9. Let V be a vector space of dimension greater than 1. Every subspace of finite codimension is a cf-element in the lattice $\mathbf{Sub}(V)$ of subspaces of V. If $\mathbf{Sub}(V) \cong \mathbf{Con}(\mathbf{A})$ for some locally finite algebra $\mathbf{A}$, then it follows from Theorem 2.5 that V/U is finite when U has finite codimension. This forces V to be a vector space over a finite field, implying that V itself is locally finite. In other words, if the congruence lattice of a vector space of dimension ≥ 1 is representable as the congruence lattice of a locally finite algebra, then the vector space itself must be locally finite. Moreover, by arguments mirroring those of Example 2.8, the vector space and the representing algebra must have the same size.

Example 2.10. Let G be a group, and let $\mathbf{L} = \mathbf{Sub}(G)$ be the lattice of subgroups of G. Then $\mathbf{L}$ is isomorphic to the congruence lattice of G considered as a G-set over itself. Such an algebra is never simultaneously locally finite and infinite, and it seems to happen frequently that the lattice $\mathbf{L}$ is not representable as the congruence lattice of any algebra that is simultaneously locally finite and infinite.

For example, if H is a nontrivial finite group and $G = H \times H$, then $\mathbf{Sub}(G)$ contains elements $H \times \{1\}, \{1\} \times H$, and the diagonal subgroup $D = \{(h,h) \mid h \in H\}$. These three subgroups pairwise join to G and pairwise meet to $\{1\}$. This is enough to show that $\mathbf{Sub}(G)$ has no homomorphism onto the 2-element chain. By Corollary 2.7, $\mathbf{Sub}(G)$ is not isomorphic to the congruence lattice of an infinite locally finite algebra.

For another example, it can be argued that if G is any nontrivial finite group satisfying $[G,G] = G$, then $\mathbf{Sub}(G)$ has no homomorphism onto a 2-element chain. (For if $\varphi\colon\ \mathbf{Sub}(G) \twoheadrightarrow \mathbf{2}$, then the largest subgroup φ maps to zero can be shown to be normal in G of prime power index. Thus, if $[G,G] = G$, then there is no such normal subgroup, so there can be no such homomorphism.) By Corollary 2.7, in this situation $\mathbf{Sub}(G)$ is not the congruence lattice of an infinite locally finite algebra.

Some subgroup lattices are not the congruence lattice of any locally finite algebra at all. For example, let $G = SO(3,\mathbb{R})$ be the special orthogonal group, viewed as the rotation group of the unit sphere in $\mathbb{R}^3$. For $\mathbf{a}$ on the unit sphere, the stabilizer $G_{\mathbf{a}}$ consists of the rotations around the axis whose direction vector is $\mathbf{a}$. If $G_{\mathbf{a}} \neq G_{\mathbf{b}}$, then $G_{\mathbf{a}} \wedge G_{\mathbf{b}} = \{1\}$ and $G_{\mathbf{a}} \vee G_{\mathbf{b}} = G$ in $\mathbf{L}$. This is enough to prove that the least complete congruence Ω such that $\mathbf{L}/\Omega$ is meet semidistributive is the total relation. Hence the cf-filter of $\mathbf{L}$ is generated by those $H \in L$ such that the interval $[H, G]$ contains a finite maximal chain. Since the interval $[G_{\mathbf{a}}, G]$ contains only three elements, namely $G_{\mathbf{a}}, G$ and the setwise stabilizer $G_{\{\mathbf{a},-\mathbf{a}\}}$ of $\{\mathbf{a}, -\mathbf{a}\}$, it follows that each $G_{\mathbf{a}}$ is in the cf-filter. Since $G_{\mathbf{a}} \wedge G_{\mathbf{b}} = \{1\}$ when $G_{\mathbf{a}} \neq G_{\mathbf{b}}$, it follows that $\{1\}$ is a cf-element. By Theorem 2.5, $\mathbf{L}$ is not the congruence lattice of a locally finite algebra.

Example 2.11. It has been shown that certain finite lattices are not representable as the congruence lattice of an infinite locally finite algebra. Here is will be shown that every finite distributive lattice is the congruence lattice of a locally finite algebra of cardinality κ for any infinite κ (and for infinitely many finite κ).

Let $\mathbf{D}$ be a finite distributive lattice, and let $\mathbf{B}$ be a Boolean lattice of cardinality κ that contains $\mathbf{D}$ as a $0,1$-sublattice. For each $x \in B$, let $c(x)$ be the least element of the sublattice $\mathbf{D}$ that lies above x. Then $c\colon B \to B$ is an increasing join homomorphism whose fixed points are the elements of D. Let $\mathbf{A} = \langle B; \vee, \wedge, c\rangle$.

$\mathbf{A}$ is locally finite, since if $\mathbf{C}$ is a subalgebra generated by a finite subset $X_0 \subseteq B$, then $\mathbf{C}$ is contained in the sublattice of $\mathbf{B}$ generated by the finite set $X_0 \cup D$. Since $\mathbf{B}$ is a locally finite lattice, $\mathbf{C}$ is finite.

Any congruence on $\mathbf{A}$ is a congruence on the Boolean lattice $\mathbf{B}$, hence is uniquely determined by the ideal I of elements congruent to 0. For the congruence to be compatible with c also, it is necessary to have $c(I) \subseteq I$, which means that I must be a principal ideal generated by an element of D. Conversely, if I is a principal ideal generated by an element $d \in D$, then the lattice congruence corresponding to I is $\mathrm{Cg}^{\mathbf{B}}(0, d) = \{(x, y) \in B^2 \mid x \vee d = y \vee d\}$. This is the kernel of the lattice endomorphism $x \mapsto x \vee d$ of $\mathbf{B}$, which is readily seen to be an endomorphism of $\mathbf{A}$ also. Hence $\mathrm{Cg}^{\mathbf{B}}(0, d)$ is a congruence of $\mathbf{A}$ when $d \in D$. Altogether this shows that the congruences of $\mathbf{A}$ are those of the form $\mathrm{Cg}^{\mathbf{B}}(0, d)$, $d \in D$. Since $\mathbf{D}$ is a sublattice, $\mathrm{Cg}(0, x) \vee \mathrm{Cg}(0, y) = \mathrm{Cg}(0, x \vee y)$ and $\mathrm{Cg}(0, x) \wedge \mathrm{Cg}(0, y) = \mathrm{Cg}(0, x \wedge y)$. This proves that $\varphi\colon \mathbf{D} \to \mathbf{Con}(\mathbf{A})\colon x \mapsto \mathrm{Cg}(0, x)$ is an isomorphism.

References

[1] R. Freese, W. A. Lampe, and W. Taylor, *Congruence lattices of algebras of fixed similarity type. I*, Pacific J. Math. **82** (1979), no. 1, 59–68.

[2] R. Freese and R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, London Mathematical Society Lecture Note Series, **125**. Cambridge University Press, Cambridge, 1987.

[3] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics, **76**, American Mathematical Society, 1988.

[4] K. A. Kearnes, *A Hamiltonian property for nilpotent algebras*, Algebra Universalis **37** (1997), no. 4, 403–421.

[5] E. W. Kiss, M. A. Valeriote, *Abelian algebras and the Hamiltonian property*, J. Pure Appl. Algebra **87** (1993), no. 1, 37–49.

[6] R. McKenzie, *Finite forbidden lattices*, Universal algebra and lattice theory (Puebla, 1982), 176–205, Lecture Notes in Math., **1004**, Springer, Berlin, 1983.

(Keith Kearnes) Department of Mathematics, University of Colorado, Boulder CO 80309, USA
E-mail address: kearnes@euclid.colorado.edu