

# SOLVABLE QUOTIENTS OF SUBDIRECT PRODUCTS OF PERFECT GROUPS ARE NILPOTENT

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ABSTRACT. We prove the statement in the title and exhibit examples of quotients of arbitrary nilpotency class. This answers a question by D. F. Holt.

## 1. INTRODUCTION

A subgroup  $S$  of a direct product  $G_1 \times \cdots \times G_n$  is said to be *subdirect* if its projection to each of the factors  $G_i$  is surjective. Subdirect products have recently been a focus of interest from combinatorial and algorithmic point of view; see, for example, [1]. They are also frequently used in computational finite group theory, notably to construct perfect groups; see [4].

In September 2017 D. F. Holt asked on the group pub forum whether a solvable quotient of a subdirect product of perfect groups is necessarily abelian. The question was re-posted on mathoverflow [3] where Holt gives some background: “This problem arose in a study of the complexity of certain algorithms for finite permutation and matrix groups. The group  $S$  in the applications is the kernel of the action of a transitive but imprimitive permutation (or matrix) group on a block system. So in that situation the  $G_i$  are all isomorphic, and  $\text{Aut}(S)$  induces a transitive action on the direct factors  $G_i$  ( $1 \leq i \leq n$ ).”

In the present note we start from the observation that every solvable quotient of a subdirect product of perfect groups is nilpotent, proved in Section 2, and then proceed to construct several examples demonstrating that the nilpotency class can be arbitrarily high. Specifically, in Section 5 we exhibit an infinite group  $G$  such that for every  $d$  there is a subdirect subgroup of  $G^{2^d}$  with a nilpotent quotient of degree  $d$ . In Section 6 we exhibit a perfect subgroup  $G$  of the special linear group  $\text{SL}_6(\mathbb{F}_4)$  such that  $G^4$  contains a subdirect subgroup with a nilpotent quotient of class 2. This example is generalized in Section 7 to a perfect subgroup  $G$  of the special linear group  $\text{SL}_{2n}(\mathbb{F}_4)$  such that  $G^{2^{n-1}}$  contains a subdirect subgroup with

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*Date:* April 9, 2018.

*2010 Mathematics Subject Classification.* Primary: 20F14, Secondary: 20E22.

*Key words and phrases.* Subdirect product, subgroup, perfect group, nilpotent.

The first two authors were supported by the National Science Foundation under Grant No. DMS 1500254.

a nilpotent quotient of class  $n - 1$ . These examples are all based on a particular subdirect product construction, which is described in Section 3, and whose lower central series is computed in Section 4.

Throughout the paper we will use the following notation. Let  $\mathbb{N}$  denote the set of positive integers. For  $n \in \mathbb{N}$ , we write  $[n]$  to denote  $\{1, \dots, n\}$ . The commutators in a group  $G$  will always assumed to be left associated, i.e.

$$[x_1, \dots, x_k] := [[x_1, x_2], \dots, x_k], \quad [N_1, \dots, N_k] := [[N_1, N_2], \dots, N_k]$$

for  $x_i \in G$  and  $N_i \trianglelefteq G$ . Furthermore, we let

$$\gamma_k(G) := \underbrace{[G, \dots, G]}_k$$

denote the  $k$ th term in the lower central series of  $G$  and, for  $N \trianglelefteq G$ , write

$$[G, {}_k N] := [G, \underbrace{N, \dots, N}_k], \quad [G, {}_0 N] := G.$$

The following two facts about commutators for a group  $G$  with a normal subgroup  $N$  will frequently be used:

- [C1]  $[\gamma_k(N), \gamma_l(N)] \leq \gamma_{k+l}(N)$ ;
- [C2]  $[G, {}_k N], \gamma_l(N) \leq [G, {}_{k+l} N]$ .

## 2. SOLVABLE QUOTIENTS MUST BE NILPOTENT

That solvable quotients of subdirect products of perfect groups are necessarily nilpotent is an easy consequence of the following more general observation.

**Theorem 2.1.** *Let  $N \trianglelefteq S \leq_{\text{sd}} G_1 \times \dots \times G_n$  such that  $N, S$  are both subdirect products of groups  $G_1, \dots, G_n$ . Then  $S/N$  is nilpotency of class at most  $n - 1$ .*

*Proof.* For  $i = 1, \dots, n$ , let  $K_i \trianglelefteq S$  be the kernel of the projection of  $S$  onto  $G_i$ . Notice that  $\bigcap_{i=1}^n K_i = 1$ . Moreover we claim that

$$K_i N = S \text{ for all } i = 1, \dots, n. \tag{1}$$

For the inclusion  $\supseteq$ , let  $s \in S$ . Since the projection of  $N$  onto  $G_i$  is onto by assumption, we have  $b \in N$  such that  $b_i = s_i$ . Then  $sb^{-1}$  is in  $K_i$  which proves (1). It now follows that

$$\gamma_n(S) = [K_1 N, \dots, K_n N] \leq [K_1, \dots, K_n] N.$$

Since  $[K_1, \dots, K_n] \leq \bigcap_{i=1}^n K_i = 1$ , we obtain  $\gamma_n(S) \leq N$ . □

*Remark 2.2.* The analogue of Theorem 2.1 holds, with exactly the same proof, in any congruence modular variety when the normal subgroup  $N$  is replaced by a congruence  $\nu$  of the subdirect product  $S$  such that the join of  $\nu$  with any projection kernel  $\rho_i$  ( $i \leq n$ ) yields the total congruence on  $S$ . In fact, then

$S/\nu$  is supernilpotent of class at most  $n - 1$ . We refer the reader to [2] for the definition of commutators of congruences and to [6] for supernilpotence and higher commutators in this general setting. It then follows that the analogue of Theorem 2.1 holds for rings,  $K$ -algebras, Lie algebras, and loops.

**Corollary 2.3.** *Let  $S \leq_{\text{sd}} G_1 \times \cdots \times G_n$  be a subdirect product of perfect groups  $G_1, \dots, G_n$ , and let  $N \trianglelefteq S$  be a normal subgroup of  $S$ . If  $S/N$  is solvable, then  $S/N$  is nilpotent of class at most  $n - 1$ .*

*Proof.* For  $i = 1, \dots, n$ , let  $N_i$  denote the projection of  $N$  on  $G_i$ . Since each  $G_i$  is perfect and  $G_i/N_i$  is solvable by assumption, we obtain  $G_i = N_i$  and the result follows from Theorem 2.1.  $\square$

### 3. A SUBDIRECT CONSTRUCTION

By Corollary 2.3, every solvable quotient of the subdirect product of two perfect groups is abelian. In what follows, we are going to show that in fact there exist subdirect powers of a perfect group  $G$  with quotients of arbitrarily large nilpotency class. To this end we introduce a construction that is based on higher commutator relations from universal algebra [6].

Let  $G$  be a group, and let  $d \in \mathbb{N}$ . We will be working in the direct product of  $2^d$  copies of  $G$  with components indexed by the power set  $\mathcal{P} := \mathcal{P}([d])$ . The set  $\mathcal{P}$  will be linearly ordered by the short-lex ordering.

Now, for any  $A \subseteq [d]$ ,  $u \in G$  and  $K \subseteq G$ , define  $\Delta_A(u) \in G^{\mathcal{P}}$  by

$$\Delta_A(u)(B) := \begin{cases} u & \text{if } A \subseteq B \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\Delta_A(K) := \{\Delta_A(u) : u \in K\}.$$

The following is an immediate consequence of the above definition:

**Lemma 3.1.** *If  $K, L \trianglelefteq G$  and  $A, B \subseteq [d]$ , then*

$$[\Delta_A(K), \Delta_B(L)] = \Delta_{A \cup B}([K, L]).$$

Now, for a normal subgroup  $N \trianglelefteq G$  and for  $k \in [d + 1]$  we define the following subset of  $G^{\mathcal{P}}$ :

$$\Gamma_k := \Gamma_k(G, N) := \prod_{|A| < k} \Delta_A([G, {}_{|A|}N]) \cdot \prod_{|A| \geq k} \Delta_A(\gamma_{|A|}(N)). \quad (2)$$

Here the factors in the products are ordered by short-lex on  $\mathcal{P}$ .

**Lemma 3.2.** *Let  $A, B \subseteq [d]$ . Then*

$$(i) \quad [[G, {}_{|A|}N], [G, {}_{|B|}N]] \leq [G, {}_{|A \cup B|}N],$$

- (ii)  $[[G, {}_{|A|}N], \gamma_{|B|}(N)] \leq [G, {}_{|A \cup B|}N]$  for  $B \neq \emptyset$  and
- (iii)  $[\gamma_{|A|}(N), \gamma_{|B|}(N)] \leq \gamma_{|A \cup B|}(N)$  for  $A, B \neq \emptyset$ .

*Proof.* Immediate from [C1] and [C2].  $\square$

**Lemma 3.3.**  $\Gamma_k(G, N)$  is a subdirect subgroup of  $G^{\mathcal{P}}$  for each  $k \in [d+1]$  and  $N \trianglelefteq G$ .

*Proof.* First notice that all factors in the definition of  $\Gamma_k$  are subgroups. That  $\Gamma_k$  projects onto each coordinate follows from the presence of  $\Delta_\emptyset([G, {}_0N]) = \Delta_\emptyset(G)$ , the diagonal of  $G^{\mathcal{P}}$ .

To prove that  $\Gamma_k$  is a subgroup, consider two generic factors  $\Delta_A(K)$  and  $\Delta_B(L)$  in its definition for  $A <_{\text{lex}} B$ . That is,  $K = [G, {}_{|A|}N]$  if  $|A| < k$  and  $K = \gamma_{|A|}(N)$  otherwise, and likewise for  $L$ . By Lemma 3.1, we obtain

$$\begin{aligned} \Delta_B(L) \cdot \Delta_A(K) &= \Delta_A(K) \cdot \Delta_B(L) \cdot [\Delta_B(L), \Delta_A(K)] \\ &= \Delta_A(K) \cdot \Delta_B(L) \cdot \Delta_{A \cup B}([K, L]). \end{aligned}$$

By Lemma 3.2, we have  $[K, L] \leq [G, {}_{|A \cup B|}N]$  if  $|A \cup B| < k$ ; further  $[K, L] \leq \gamma_{|A \cup B|}(N)$  if  $|A \cup B| \geq k$ . Hence  $\Delta_{A \cup B}([K, L])$  is contained in another factor of the product defining  $\Gamma_k$ . Since  $B \leq_{\text{lex}} A \cup B$ , the result follows.  $\square$

*Remark 3.4.* In reference to Holt's motivation, we note that the automorphism group of  $S := \Gamma_1$  acts transitively on the index set  $\mathcal{P}$ . More precisely, for  $i \in [d]$ , the permutation on  $\mathcal{P}$  that maps a subset  $A$  of  $[d]$  to its symmetric difference with  $\{i\}$  induces an automorphism  $r_i$  on  $S$ . This can be easily seen by the action of  $r_i$  on the generators  $\Delta_A(u)$  of  $S$ :  $r_i(\Delta_A(u)) = \Delta_A(u)$  if  $i \notin A$ ; else  $r_i(\Delta_A(u)) = \Delta_{A \setminus \{i\}}(u) \cdot \Delta_A(u)^{-1}$ . Then  $\langle r_1, \dots, r_d \rangle \leq \text{Aut}(S)$  is isomorphic to  $\mathbb{Z}_2^d$  and acts regularly on the set of direct factors of  $S$ .

#### 4. THE LOWER CENTRAL SERIES

Continuing with the construction from the previous section, we now let  $S := \Gamma_1$  and determine its lower central series:

**Lemma 4.1.** *If  $G' = G$ , then  $\gamma_k(S) = \Gamma_k$  for all  $k \in [d+1]$ .*

*Proof.* We need to show that

$$[\Gamma_k, \Gamma_1] = \Gamma_{k+1} \quad \text{for } k \in [d]. \quad (3)$$

For the inclusion  $\leq$  it is sufficient to prove that

$$[\Delta_A(K), \Delta_B(L)] = \Delta_{A \cup B}([K, L]) \leq \Gamma_{k+1}, \quad (4)$$

where

$$K = \begin{cases} [G, {}_{|A|}N] & \text{if } |A| < k, \\ \gamma_{{}_{|A|}}(N) & \text{otherwise,} \end{cases} \quad L = \begin{cases} G & \text{if } B = \emptyset, \\ \gamma_{{}_{|B|}}(N) & \text{otherwise.} \end{cases}$$

Note that (4) follows from the claim that

$$[K, L] \leq \begin{cases} [G, {}_{|A \cup B|}N] & \text{if } |A \cup B| < k + 1, \\ \gamma_{{}_{|A \cup B|}}(N) & \text{otherwise.} \end{cases} \quad (5)$$

The assertion for  $|A \cup B| < k + 1$  follows from Lemma 3.2 except when  $|A| = k$  and  $B \subseteq A$ . In this case we use [C2] to obtain

$$[K, L] = [\gamma_{{}_{|A|}}(N), L] \leq [G, \gamma_{{}_{|A|}}(N)] \leq [G, {}_{|A|}N] = [G, {}_{|A \cup B|}N].$$

Claim (5) for  $|A \cup B| \geq k + 1$  is also immediate from Lemma 3.2 and the observation that  $[G, {}_{|A \cup B|}N] \leq \gamma_{{}_{|A \cup B|}}(N)$  for  $A \cup B \neq \emptyset$ . Hence (5), (4), and the inclusion  $\subseteq$  of (3) are proved.

For the inclusion  $\supseteq$  of (3) we show that

$$\Delta_A(K) \subseteq [\Gamma_k, \Gamma_1], \quad (6)$$

where

$$K = \begin{cases} [G, {}_{|A|}N] & \text{if } |A| < k + 1 \\ \gamma_{{}_{|A|}}(N) & \text{if } |A| \geq k + 1. \end{cases}$$

Now, if  $A = \emptyset$ , the assumption that  $G = G'$  yields

$$\Delta_\emptyset(K) = \Delta_\emptyset(G) = \Delta_\emptyset([G, G]) = [\Delta_\emptyset(G), \Delta_\emptyset(G)] \subseteq [\Gamma_k, \Gamma_1].$$

Assume now that  $A \neq \emptyset$ , and write it as  $A = B \cup C$ , where  $|B| = |A| - 1$  and  $|C| = 1$ . If  $|A| < k + 1$ , then  $|B| < k$  and by Lemma 3.1

$$\Delta_A(K) = \Delta_{B \cup C}([G, {}_{|B|+|C|}N]) = [\Delta_B([G, {}_{|B|}N]), \Delta_C(N)] \subseteq [\Gamma_k, \Gamma_1].$$

Likewise, if  $|A| \geq k + 1$ , then  $|B| \geq k$  and

$$\Delta_A(K) = \Delta_{B \cup C}(\gamma_{{}_{|B|+|C|}}(N)) = [\Delta_B(\gamma_{{}_{|B|}}(N)), \Delta_C(N)] \subseteq [\Gamma_k, \Gamma_1].$$

This completes the proof of (6), the inclusion  $\supseteq$  of (3) and the lemma.  $\square$

We can now prove our main result:

**Theorem 4.2.** *Let  $G$  be a perfect group,  $d \in \mathbb{N}$ , and  $N$  a normal subgroup such that  $\gamma_d(N) > [G, {}_dN]$ . Then  $S = \Gamma_1(G, N)$  is a subdirect subgroup of  $G^{2^d}$  which has a nilpotent quotient of class  $d$ .*

*Proof.* That  $S$  is subdirect was proved in Lemma 3.3. Let  $u \in \gamma_d(N) \setminus [G, {}_dN]$ . By Lemma 4.1, we have

$$\Delta_{[d]}(u) \in \Gamma_d \setminus \Gamma_{d+1} = \gamma_d(S) \setminus \gamma_{d+1}(S).$$

Hence  $\gamma_d(S) > \gamma_{d+1}(S)$ , and it follows that  $S/\gamma_{d+1}(S)$  is  $d$ -nilpotent.  $\square$

## 5. AN INFINITE EXAMPLE

In this section we demonstrate the existence of a perfect group  $G$  satisfying the assumptions of Theorem 4.2. More precisely, we will construct an infinite group  $G$  with a normal subgroup  $N$  such that

$$\gamma_d(N) > [G, {}_dN] \text{ for all } d \in \mathbb{N}. \quad (7)$$

Let  $X$  be a countably infinite alphabet with elements

$$X := \{x_{i,w} : i \in \mathbb{N}, w \in \{0,1\}^*\} \cup \{y_i : i \in \mathbb{N}\}.$$

Here  $\{0,1\}^*$  denotes the set of all words over  $\{0,1\}$  including the empty word  $\epsilon$ . Let  $F$  be the free group on  $X$ . The derived subgroup  $F'$  is free by Nielsen's Theorem, and it is routine to see that it has a free basis  $Z$  which contains all the commutators  $[u, v]$  with  $u, v \in X$ ,  $u \neq v$ . For  $i \in \mathbb{N}$ ,  $w \in \{0,1\}^*$  set

$$h(x_{i,w}) := [x_{i,w0}, x_{i,w1}] \in Z,$$

and extend  $h$  to a bijection  $h : X \rightarrow Z$ , and then to an isomorphism  $h : F \rightarrow F'$ .

Now let  $F_i := F$  for  $i \in \mathbb{N}$ , and let  $G$  be the limit of the directed system

$$F_1 \xrightarrow{h} F_2 \xrightarrow{h} F_3 \xrightarrow{h} \dots$$

Thus, there exist embeddings  $h_i : F_i \rightarrow G$  ( $i \in \mathbb{N}$ ) such that  $h_i = h_{i+1}h$ ,  $h_1(F_1) \leq h_2(F_2) \leq \dots$  and  $G = \bigcup_{i \in \mathbb{N}} h_i(F_i)$ .

The group  $G$  is perfect, since  $h_i(F_i) = h_{i+1}(F_{i+1})'$ . Also, from  $h_1(F_1) = h_i(F_i)^{(i-1)}$ , it follows that  $N := h_1(F_1) \trianglelefteq G$ . We claim that (7) holds for this  $G$  and  $N$ .

Suppose to the contrary that  $\gamma_d(N) = [G, {}_dN]$  for some  $d$ . Consider the element

$$u := [x_{1,\epsilon}, x_{2,\epsilon}, \dots, x_{d,\epsilon}] \in \gamma_d(F_1).$$

Then  $h_1(u) \in \gamma_d(N)$  and hence  $h_1(u) \in [G, {}_dN]$ . It follows that

$$h_1(u) = v_1 v_2 \dots v_k,$$

where

$$v_i = c_i^{-1} [g_i, n_{i1}, \dots, n_{id}]^{e_i} c_i$$

for  $e_i \in \{-1, 1\}$ ,  $c_i, g_i \in G$ ,  $n_{ij} \in N$  for all  $i \leq k$ ,  $j \leq d$ . From  $G = \bigcup_{t \in \mathbb{N}} h_t(F_t)$  it follows that there exists  $t \in \mathbb{N}$  such that  $c_i, g_i \in h_t(F_t)$  for all  $i = 1, \dots, k$ . Therefore we have

$$h_1(u) \in [h_t(F_t), {}_d h_1(F_1)].$$

Since  $h_1 = h_t h^{t-1}$  and  $h_t : F_t \rightarrow G$  is an injection, this implies

$$h^{t-1}(u) \in [F_t, {}_d h^{t-1}(F_1)].$$

Recall  $F_1 = F_t = F$  and  $h^{t-1}(F) = F^{(t-1)}$  to obtain

$$h^{t-1}(u) \in [F, {}_d F^{(t-1)}]. \quad (8)$$

Note that

$$h^{t-1}(x_{i,\epsilon}) = h^{t-2}([x_{i,0}, x_{i,1}]) = h^{t-3}([ [x_{i,00}, x_{i,01}], [x_{i,10}, x_{i,11}] ]) = \dots$$

generates  $F^{(t-1)}$  as a fully invariant subgroup of  $F$ . Hence

$$h^{t-1}(u) = [h^{t-1}(x_{1,\epsilon}), \dots, h^{t-1}(x_{d,\epsilon})]$$

generates  $\gamma_d(F^{(t-1)})$  as a fully invariant subgroup. Since  $[F, {}_dF^{(t-1)}]$  is also fully invariant, (8) implies the inclusion  $\leq$  in

$$\gamma_d(F^{(t-1)}) = [F, {}_dF^{(t-1)}].$$

The converse inclusion  $\geq$  is trivial. Since  $F$  is free of countable rank, this implies that

$$\gamma_d(H^{(t-1)}) = [H, {}_dH^{(t-1)}]$$

for *all* countable (not necessarily perfect) groups  $H$ . But this is false, one counterexample being the group of upper unitriangular  $n \times n$  matrices over any field for sufficiently large  $n$ . This contradiction establishes Claim (7).

## 6. A FINITE EXAMPLE

In this section we exhibit a finite group  $G$  with a normal subgroup  $N$  such that  $[N, N] > [G, N, N]$ .

Let  $\mathbb{F} := \mathbb{F}_4$  be the field of order 4, and let  $\alpha \in \mathbb{F}$  be a primitive element; recall that  $\alpha^2 = \alpha + 1$ . Recall that the special linear group  $S := \mathrm{SL}_2(\mathbb{F})$  is perfect [5, 3.2.8]; let  $I$  denote its identity. Also let

$$T := \{A \in \mathbb{F}^{2 \times 2} : \mathrm{tr}(A) = 0\},$$

the set of all matrices of trace 0.

In what follows we will work in the special linear group  $\mathrm{SL}_6(\mathbb{F})$ . Its elements will be considered and written as  $3 \times 3$  block matrices over  $\mathbb{F}^{2 \times 2}$ . First define the following three subgroups of  $\mathrm{SL}_6(\mathbb{F})$ :

$$H := \left\{ \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} : A \in S \right\},$$

$$M := \left\{ \begin{pmatrix} I & B & D \\ & I & C \\ & & I \end{pmatrix} : B, C, D \in \mathbb{F}^{2 \times 2}, \mathrm{tr}(B) = \mathrm{tr}(C) = 0 \right\},$$

$$N := \left\{ \begin{pmatrix} I & bI & D \\ & I & cI \\ & & I \end{pmatrix} : b, c \in \mathbb{F}, D \in \mathbb{F}^{2 \times 2} \right\},$$

and then set

$$G := HM.$$

**Lemma 6.1.** *The following hold for  $G$ ,  $N$  and  $M$  as defined above:*

- (i)  $M \trianglelefteq G \leq \mathrm{SL}_6(\mathbb{F})$ .
- (ii)  $G$  is perfect.
- (iii)  $N \trianglelefteq G$ .
- (iv)  $[N, N] > [G, N, N]$ .

In the proof of the above results, we will make use of the following two technical observations.

**Lemma 6.2.** *The set  $T$ , considered as an  $\mathbb{F}$ -module, is spanned by the set*

$$U := \{-B + B^A : B \in T, A \in S\}.$$

*Proof.* For every  $x \in \mathbb{F}$  we have

$$\begin{pmatrix} x & x^2 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U,$$

and hence

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha & \alpha^2 \\ 0 & \alpha \end{pmatrix} \in \mathbb{F}U.$$

By symmetry  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}U$ , and also  $I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}U$ . Since  $T$  is spanned by  $I$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , the assertion follows.  $\square$

**Lemma 6.3.** *The  $\mathbb{F}$ -module spanned by the set*

$$U := \{BC : B, C \in T\}$$

*is the entire  $\mathbb{F}^{2 \times 2}$ .*

*Proof.* The subset

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cdot \left\{ I, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

of  $U$  contains the natural basis of  $\mathbb{F}^{2 \times 2}$ .  $\square$

*Proof of Lemma 6.1.* (i) For

$$X := \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} \in H \text{ and } Y := \begin{pmatrix} I & B & D \\ & I & C \\ & & I \end{pmatrix} \in M \quad (9)$$

we have

$$Y^X = \begin{pmatrix} I & B^A & D^A \\ & I & C^A \\ & & I \end{pmatrix}.$$

From  $\mathrm{tr}(B^A) = \mathrm{tr}(B) = 0$  and  $\mathrm{tr}(C^A) = \mathrm{tr}(C) = 0$  it follows that  $Y^X \in M$ . Thus,  $H$  normalizes  $M$ , and both assertions follow.



(ii) Since  $H \cong S$  is perfect, we have

$$[G, G] = [HM, HM] = [H, H][H, M][M, M] = H[H, M]M'.$$

So it suffices to show that

$$[H, M] = M. \quad (10)$$

Now, for  $X \in H, Y \in M$  as in (9), we have

$$Y^{-1} = \begin{pmatrix} I & -B & BC - D \\ & I & -C \\ & & I \end{pmatrix}$$

and hence

$$[Y, X] = Y^{-1}Y^X = \begin{pmatrix} I & -B + B^A & D^A - BC^A + BC - D \\ & I & -C + C^A \\ & & I \end{pmatrix}. \quad (11)$$

Using Lemma 6.2 and (11) twice, first with  $C = D = 0$ , and then with  $B = C = 0$ , we see that

$$\begin{pmatrix} I & P & 0 \\ & I & 0 \\ & & I \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ & I & Q \\ & & I \end{pmatrix} \in [H, M] \text{ for all } P, Q \in T.$$

Multiplying these two matrices in both orders yields

$$\begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix}, \begin{pmatrix} I & P & PQ \\ & I & Q \\ & & I \end{pmatrix} \in [H, M] \text{ for all } P, Q \in T. \quad (12)$$

Hence also

$$\begin{pmatrix} I & 0 & PQ \\ & I & 0 \\ & & I \end{pmatrix} = \begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix}^{-1} \begin{pmatrix} I & P & PQ \\ & I & Q \\ & & I \end{pmatrix} \in [H, M]. \quad (13)$$

Using Lemma 6.3, it follows that

$$\begin{pmatrix} I & 0 & R \\ & I & 0 \\ & & I \end{pmatrix} \in [H, M] \text{ for all } R \in \mathbb{F}^{2 \times 2}. \quad (14)$$

Multiplying the first matrix from (12) together with (14) yields

$$\begin{pmatrix} I & P & R \\ & I & Q \\ & & I \end{pmatrix} = \begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix} \begin{pmatrix} I & 0 & R \\ & I & 0 \\ & & I \end{pmatrix} \in [H, M]$$

for all  $P, Q \in T, R \in \mathbb{F}^{2 \times 2}$ , proving Claim (10) and part (ii) of the lemma.

(iii) Again, let  $X \in H$  and  $Y \in M$  be as in (9), and also let

$$Z := \begin{pmatrix} I & bI & E \\ & I & cI \\ & & I \end{pmatrix} \in N.$$

Then

$$Z^X = \begin{pmatrix} I & bI & E^A \\ & I & cI \\ & & I \end{pmatrix} \text{ and } Z^Y = \begin{pmatrix} I & bI & bC - cB + E \\ & I & cI \\ & & I \end{pmatrix}$$

are both in  $N$ . Hence  $N \trianglelefteq G$ .

(iv) To compute  $[G, N]$ , note that, for  $X \in H, Y \in M, Z \in N$  as above, we have:

$$\begin{aligned} [Z, X] &= Z^{-1}Z^X = \begin{pmatrix} I & 0 & -E + E^A \\ & I & 0 \\ & & I \end{pmatrix}, \\ [Z, Y] &= Z^{-1}Z^Y = \begin{pmatrix} I & 0 & bC - cB \\ & I & 0 \\ & & I \end{pmatrix}. \end{aligned} \tag{15}$$

Recalling Lemma 6.2 it follows that

$$[G, N] = \left\{ \begin{pmatrix} I & 0 & P \\ & I & 0 \\ & & I \end{pmatrix} : P \in T \right\}. \tag{16}$$

From here it is immediate that  $[G, N, N]$  is trivial. On the other hand, for  $Y \in N$  with  $B = b'I, C = c'I$  in (15) we obtain

$$[Z, Y] = \begin{pmatrix} I & 0 & (bc' - b'c)I \\ & I & 0 \\ & & I \end{pmatrix}.$$

Hence

$$[N, N] = \left\{ \begin{pmatrix} I & 0 & dI \\ & I & 0 \\ & & I \end{pmatrix} : d \in \mathbb{F} \right\}$$

is strictly bigger than  $[G, N, N]$ .  $\square$

## 7. A FAMILY OF FINITE EXAMPLES

The construction from the previous section can be readily adapted to yield, for every  $d \in \mathbb{N}$ , a finite perfect group  $G$  and a normal subgroup  $N$  such that  $\gamma_d(N) > [G, {}_dN]$ . We just give an outline.

Let  $n \in \mathbb{N}$  be arbitrary. The field  $\mathbb{F}$ , group  $S$  and set  $T$  will be as in the previous section. This time we will work in the group  $\mathrm{SL}_{2n}(\mathbb{F})$ , which we will treat as consisting of  $n \times n$  block-matrices with  $2 \times 2$  entries.

For a matrix  $A \in S$  we will write  $\Delta(A)$  to denote the block diagonal matrix having  $n$  copies of  $A$  along the main diagonal. Also, for arbitrary  $A \in \mathbb{F}^{2 \times 2}$  and  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , we will use  $E_{ij}(A)$  to denote the matrix which has the identity blocks on the main diagonal, has  $(i, j)$  block equal to  $A$ , and has zero blocks elsewhere.

The basic ingredients for our construction are the following subgroups of  $\mathrm{SL}_{2n}(\mathbb{F})$ . Let

$$H := \{\Delta(A) : A \in S\} \cong S.$$

The second subgroup  $M$  consists of all block upper unitriangular matrices with arbitrary matrices of trace 0 on the super-diagonal, i.e.,

$$M := \{(A_{ij}) : A_{ii} = I, A_{i,i+1} \in T, A_{ij} \in \mathbb{F}^{2 \times 2} (j - i > 1), A_{ij} = 0 (i < j)\}.$$

The third subgroup  $N$  consists of those elements of  $M$  for which all the blocks on the super-diagonal belong to the center of  $S$ , i.e,

$$N := \{(A_{ij}) \in M : A_{i,i+1} = a_i I (a_i \in \mathbb{F})\}.$$

Finally, we let

$$G := HM.$$

**Lemma 7.1.** *The following hold for  $G$ ,  $N$  and  $M$  as defined above:*

- (i)  $M \trianglelefteq G \leq \mathrm{SL}_{2n}(\mathbb{F})$ .
- (ii)  $G$  is perfect.
- (iii)  $N \trianglelefteq G$ .
- (iv)  $\gamma_d(N) > [G, {}_d N]$  for  $d < n$ .

*Proof.* Items (i) and (iii) follow exactly as for Lemma 6.1. For proving (ii), we again just need

$$[H, M] = M.$$

For this, let  $A \in S$ ,  $B \in T$  and  $E_{i,i+1}(B) \in M$  for  $1 \leq i \leq n - 1$ , and note that

$$[E_{i,i+1}(B), \Delta(A)] = E_{i,i+1}(-B + B^A).$$

By Lemma 6.2 it follows that

$$\{E_{i,i+1}(P) : P \in T, 1 \leq i \leq n - 1\} \subseteq [H, M]. \quad (17)$$

As in (13) we obtain

$$[E_{i,i+1}(P), E_{i+1,i+2}(Q)] = E_{i,i+2}(PQ).$$

By Lemma 6.3 it follows that

$$\{E_{i,i+2}(R) : R \in \mathbb{F}^{2 \times 2}, 1 \leq i \leq n - 2\} \subseteq [H, M].$$

A straightforward inductive argument now shows that

$$\{E_{ij}(R) : R \in \mathbb{F}^{2 \times 2}, j - i > 1\} \subseteq [H, M]. \quad (18)$$

From (17) and (18) it easily follows that  $[H, M] = M$ , as claimed.

For (iv), we obtain as in (16)

$$[G, N] = \{(A_{ij}) \in N : A_{i,i+1} = 0, A_{i,i+2} \in T\}.$$

Assuming  $n > d$ , from this it is easy to see that

$$[G, {}_dN] = \{(A_{ij}) : A_{ij} = 0 \ (0 < j - i \leq d), A_{i,i+d+1} \in T\}.$$

Finally,

$$\gamma_d(N) = \{(A_{ij}) \in N : A_{ij} = 0 \ (0 < j - i \leq d - 1), A_{i,i+d} = a_i I \ (a_i \in \mathbb{F})\},$$

and therefore  $\gamma_d(N) > [G, {}_dN]$ , as required.  $\square$

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