Project 4: Area

Due: December 6, 1999

In this session, we will estimate areas of some regions in the plane. We will use tools in Maple's ''student'' package to inscribe rectangles below curves and then compute the sum of the areas of the rectangles.

Some Commands From the Student Package

Maple's ''student'' package has a number of tools especially designed for use while learning calculus (and other subjects). To have access to these tools we have to load the package. Since this is the first time we have done this, we will load the package with the appropriate command followed by a **semicolon**. Then you will see a list of the tools available. If you ever load the package again you will probably want to use a **colon,** which will suppress the list of tools.

```
> with(student);
Warning, new definition for D
```

[D, *Diff*, *Doubleint*, *Int*, *Limit*, *Lineint*, *Product*, *Sum*, *Tripleint*, *changevar*, *combine*, *completesquare*, *distance*, *equate*, *extrema*, *integrand*, *intercept*, *intparts*, *isolate*, *leftbox*, *leftsum*, *makeproc*, *maximize*, *middlebox*, *middlesum*, *midpoint*, *minimize*, *powsubs*, *rightbox*, *rightsum*, *showtangent*, *simpson*, *slope*, *summand*, *trapezoid*, *value*]

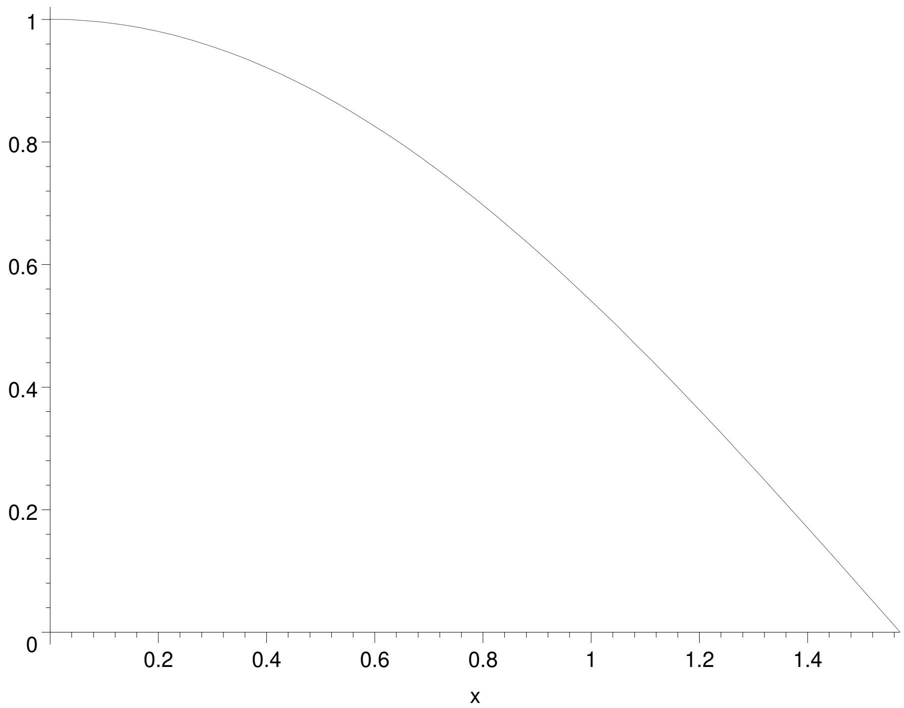
If you ''restart'' below you will have to load the student package again before continuing.

In our first area exercise we will estimate the area that lies below one half of an arch of the cosine curve. First let's have a look:

```
> f:=x->cos(x);
```

$$f := \cos$$

```
> plot(f(x),x=0..Pi/2);
```

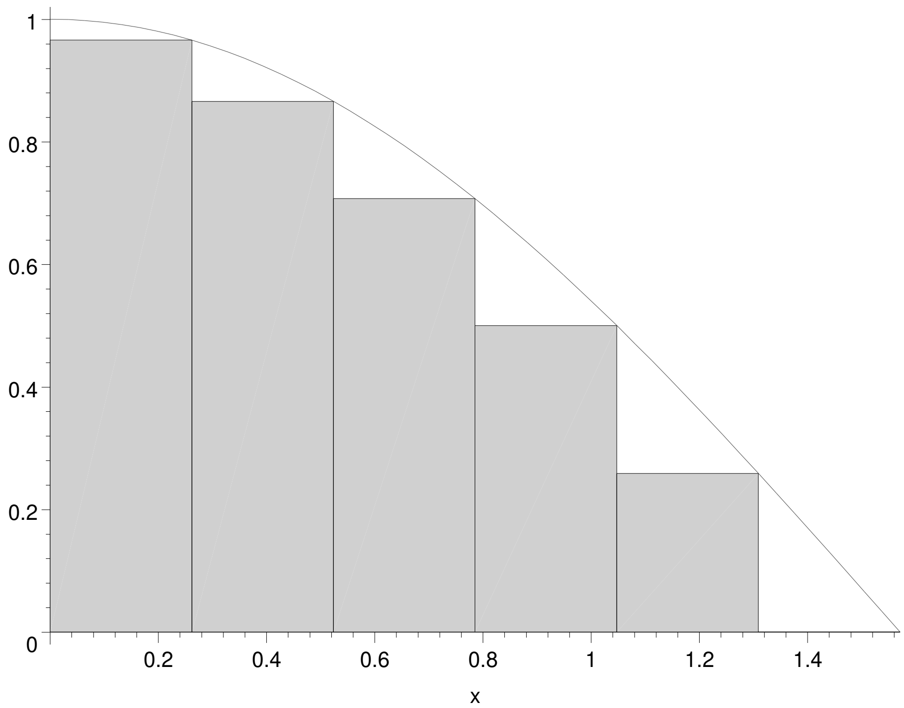


We will use the ‘‘rightbox’’ command to inscribe six rectangles below the curve.

```
> n:=6;
```

$$n := 6$$

```
> rightbox(f(x),x=0..Pi/2,n);
```




```
> L[n]:=rightsum(f(x),x=0..Pi/2,n);
```

$$L_6 := \frac{1}{12}\pi\left(\sum_{i=1}^{6}\cos\left(\frac{1}{12}i\pi\right)\right)$$

```
> value(L[n]);
```

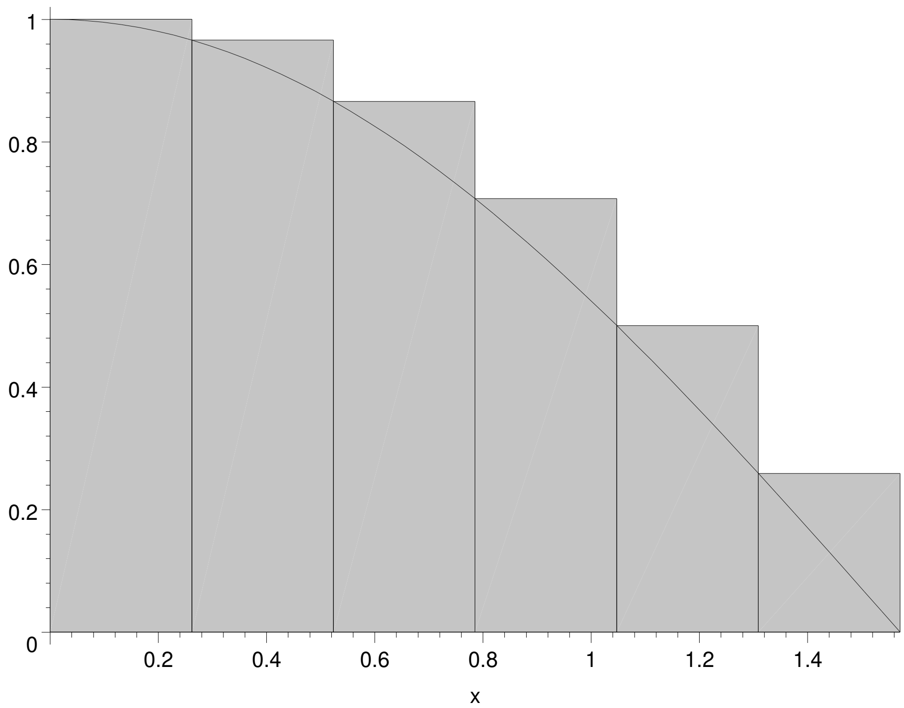
$$\frac{1}{12}\pi\left(\frac{1}{4}\sqrt{6}\left(1+\frac{1}{3}\sqrt{3}\right)+\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{2}+\frac{1}{2}+\frac{1}{4}\sqrt{6}\left(1-\frac{1}{3}\sqrt{3}\right)\right)$$

```
> Lower[n]:=evalf(L[n]);
```

$$Lower_6 := .8633821941$$

The ''rightbox'' command with $n = 6$ draws six boxes of equal base length between the graph of the function and the x-axis in such a way that the boxes touch the graph on the right side. (The commands ''leftbox'' and ''middlebox'' are similar, but the boxes touch on the left or middle.) The ''rightsum'' command adds the areas of the rightboxes. Note that in this example the function is decreasing, so all rightboxes lie below the curve. This shows that our area estimate is a *lower estimate*. If we wanted an upper estimate on the area, then since the function is decreasing we should use the leftbox/leftsum commands.

```
> leftbox(f(x),x=0..Pi/2,n,shading=plum,color=blue);
```




```
> U[n]:=leftsum(f(x),x=0..Pi/2,n);
```

$$U_6 := \frac{1}{12}\pi\left(\sum_{i=0}^{5}\cos\left(\frac{1}{12}i\pi\right)\right)$$

```
> value(U[n]);
```

$$\frac{1}{12}\pi\left(\frac{3}{2}+\frac{1}{4}\sqrt{6}\left(1+\frac{1}{3}\sqrt{3}\right)+\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{2}+\frac{1}{4}\sqrt{6}\left(1-\frac{1}{3}\sqrt{3}\right)\right)$$

```
> Upper[n]:=evalf(U[n]);
```

$$Upper_6 := 1.125181582$$

This shows that the area between the curve and the x-axis is between $Lower_6 := .8633821941$ and $Upper_6 := 1.125181582$. We can improve the estimate by subdividing the interval into finer pieces, and adding the areas of more (but thinner) rectangles.

```
> n:=16;
```

$$n := 16$$

```
> L[n]:=rightsum(f(x),x=0..Pi/2,n);
```

$$L_{16} := \frac{1}{32}\pi\left(\sum_{i=1}^{16}\cos\left(\frac{1}{32}i\pi\right)\right)$$

```
> Lower[n]:=evalf(L[n]);
```

$$Lower_{16} := .9501092953$$

```
> U[n]:=leftsum(f(x),x=0..Pi/2,n);
```

$$U_{16} := \frac{1}{32}\pi\left(\sum_{i=0}^{15}\cos\left(\frac{1}{32}i\pi\right)\right)$$

```
> Upper[n]:=evalf(U[n]);
```

$$Upper_{16} := 1.048284066$$

Exercises

1. Estimate the area between the graph of the cosine function and the x-axis on the interval $\left[0\,..\,\frac{\pi}{2}\right]$ using $n = 6$ and $n = 16$, but using the ''middlesum'' command instead of rightsum or leftsum.
2. The student package has a command called ''trapezoid'', which computes the average of leftsum and rightsum. Estimate the area in Exercise 1 using the trapezoid command with $n = 16$.

Estimating π

The area of a circle of radius r is πr^2. One can estimate π by estimating the area of a circle of radius 1. In fact, it is easier to estimate the area of the quarter circle in the first quadrant, so the next exercise asks you to do that and then multiply your answer by 4.

Exercise

3. Use commands from above to estimate π to within 0.1.

Areas Under Curves

You will be asked to estimate the area bounded between the graph of a function and the x-axis. You will be asked to produce a definite lower bound and a definite upper bound. This requires you to sum rectangles which lie completely below the curve (for the lower bound) and sum rectangles that lie completely above the curve (for the upper bound). Note that rightsum is good for summing the area of rectangles below a decreasing function, or above an increasing function. Thus, to complete the exercises successfully, you must first determine the intervals of increase/decrease.

Exercises

4. Estimate the area of the circular cap defined by the part of the function $f := x \rightarrow \sqrt{1 - x^2} - \frac{\sqrt{3}}{2}$ that lies above the x-axis. Choose n so that the difference

between the upper and lower estimates is no greater than 0.01.

5. Estimate the area below $f := x \to x^{\left(\frac{1}{3}\right)} (1 - x)^{\left(\frac{2}{3}\right)}$ and above the x-axis sufficiently accurately so that the difference betwen your upper estimate and your lower estimate is no more than 0.05.

6. Estimate the area that lies above the graph of the function $y = x^2$ and below the graph of the circle defined by $x^2 + y^2 = 1$. Your estimate should be accurate to within 0.04.

Areas Bounded by Discontinuous Functions

Problems arise when trying to calculate the area below the graph of a discontinuous function. If the function is unbounded, we cannot use rectangles to produce an upper estimate. If the function oscillates too much, there is a different kind of problem.

Exercises

7. Write a loop that uses rightsum for producing lower estimates of the area below $f := x \to \frac{1}{x^2}$ and above the x-axis on the interval $[0 .. 1]$. Use n rectangles, for $n = 5$ to $n = 100$ in steps of size 5. Do you see any pattern? What do you think the area might be?

8. Repeat Exercise 7 with the function $f := x \to \frac{1}{\sqrt{x}}$. Answer the same questions.

9. View the output that results when you use rightbox, leftbox and middlebox to approximate the area below $f := x \to 1 + \sin\left(\frac{10}{x}\right)$ on the interval $[-.1 .. 1]$. Write a few sentences explaining what is difficult about estimating the area below a curve that has oscillation discontinuities.