Project 1: Limits

Due: September 27, 1999

In this session, we will use limits to investigate geometric properties of curves in the plane.

Maple's 'limit' Command

To find the limit of a function at a point we use Maple's 'limit' command:

```
> F:=x -> x^2;
> limit(F(x),x=3);
```

Remember: $\lim_{x \to a} F(x)$ does NOT mean the same thing as $F(a)$!

```
> G:=x -> (x^2-9)/(x-3);
> G(3);
> limit(G(x),x=3);
```

Exercises

Plot $y = f(x)$ on an interval containing $x = a$ and then find $\lim_{x \to a} f(x)$ in each of the following problems.

1. $f := x \to \frac{\sqrt{x^2 + 2x + 3} - 6}{x - 1}$, $a = 1$.
2. $f := x \to \frac{1 - \cos(x)}{x^2}$, $a = 0$.
3. $f := x \to \frac{\sin(x)}{1 - \cos(x)}$, $a = 0$.
4. $f := x \to \frac{1}{\mathbf{e}^x - x - 1}$, $a = 0$.

Maple has the capability to calculate one-sided limits, too.

```
> limit(abs(x)/x,x=0,right);
> limit(abs(x)/x,x=0,left);
> limit(abs(x)/x,x=0);
```

Exercise

5. Find the limit of $f(x) = x^x$ as x approaches zero from the right.

Can you explain Maple's answer to the next problem?

```
> limit(sin(1/x),x=0);
```

Horizontal Asymptotes

It is clear that the function $f(x) = 1 + \frac{1}{x}$ has a horizontal asymptote, and that the asymptote is the line $y = 1$. Maple can be used to determine the horizontal asymptote by taking a limit at infinity or -infinity.

```
> limit(1+1/x,x=infinity);
> plot([1+1/x,1],x=0..infinity,color=[red,blue]);
```

Exercises

Plot $y = f(x)$ on an interval containing infinity, compute $\lim_{x \to \infty} f(x)$, and then plot $y = f(x)$ together with the horizontal asymptote (if it exists) on the same coordinate system.

6. $f := x \to x^{\left(\frac{1}{x}\right)}$.
7. $f := x \to \left(1 + \frac{1}{x}\right)^x$.
8. $f := x \to \mathbf{e}^{(-x)} \tan(x)$.

9. $f := x \rightarrow (\mathbf{e}^{(-x)} + 1)\sin(x)$.

10. $f := \ln(x)$.

Tangent Lines

The slope of the tangent line to the curve $y = \mathrm{f}(x)$ at the point $x = a$ can be determined by taking the limit of the slope of secant lines.

```
> f:= x -> x^3 - 2*x + 1;
> plot(f,-1..2);
```

Let's compute the tangent to this curve at the point $a = 1$. First, the slope of the secant line between the points $[1, \mathrm{f}(1)]$ and $[x, \mathrm{f}(x)]$ is given by:

```
> slope := x -> (f(x)-f(1))/(x-1);
```

Let's plot a couple of secant lines first.

```
> plot([f(x),slope(1.3)*(x-1.3)+f(1.3),slope(1.5)*(x-1.5)+f(1.5)],x=-1..2,color=[red,blu
  e,brown]);
```

Now let's find the slope of the tangent by taking a limit.

```
> s:=limit(slope(x),x=1);
> plot([f(x),s*(x-1)+f(1)],x=-1..2,color=[red,blue]);
```

Exercises

Find the line tangent to the graph of $y = \mathrm{f}(x)$ at $x = a$ (if it exists).

11. $f := x \rightarrow \sqrt{x^2 + 3}, a = 1$.

12. $f := x \rightarrow \sin\left(x + \frac{\pi}{3}\right), a = 0$.

13. $f := x \rightarrow x^2 \sin\left(\frac{1}{x}\right), a = 0$.

14. $f := x \rightarrow x \sin\left(\frac{1}{x}\right), a = 0$.

15. Plot the function $y = x^4 - 5x^2 + x + 5$ between $x = -3$ and $x = 3$. Try to determine a way to find the equation of a line that is tangent to this curve at two different points.

Polynomial Approximations to Curves

The tangent line is a linear approximation to a curve. We can use limits to approximate curves by higher-degree polynomials. For example, one can show that the line tangent to $\mathrm{f}(x) = \mathbf{e}^x$ at $x = 0$ is $1 + x$. Therefore $\mathbf{e}^x$ is approximately equal to $1 + x$ when x is near zero. A quadratic approximation to $\mathbf{e}^x$ near zero should have the form $1 + x + c\,x^2$ for some coefficient c. Solving $\mathbf{e}^x = 1 + x + c\,x^2$ for c, we find that c should be approximately $\frac{\mathbf{e}^x - 1 - x}{x^2}$ when x is near zero. Therefore, it is reasonable to suppose that the best value for c is the limit:

```
> c:= limit((exp(x)-1-x)/x^2,x=0);
```

Let's check this conclusion by plotting the function $y = \mathbf{e}^x$ on the same coordinate system as our approximation $1 + x + \frac{x^2}{2}$.

```
> plot([exp(x), 1+x+(1/2)*x^2],x=-1..1,color=[red,blue]);
```

Exercises

Find the quadratic approximation of the function $\mathrm{f}(x)$ near $x = 0$, and plot the function $\mathrm{f}(x)$ and its quadratic approximation on the same coordinate system.

16. $f := x \rightarrow \sin\left(x + \frac{\pi}{3}\right), a = 0$.

17. $f := x \rightarrow \sqrt{x + 1}, a = 0$.

18. $f := x \rightarrow (x + 1)^{(x+1)}, a = 0$.