

Definability versus Undefinability

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 - ⑤ any finite union of (open, closed, half open, infinite) intervals whose endpoints are algebraic numbers.

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This expresses “ $2z = u$ and $u = 2xy$ ”, which reduces to $2z = 2xy$ or $z = xy$.

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We saw that multiplication in $\mathbb{R}$ is definable from addition ($A(x, y) = x + y$), and squaring ($s(x) = x^2$). Do we really need both of these operations to define multiplication? Might it be that multiplication is definable from addition alone? Might it be that multiplication is definable from squaring alone?

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