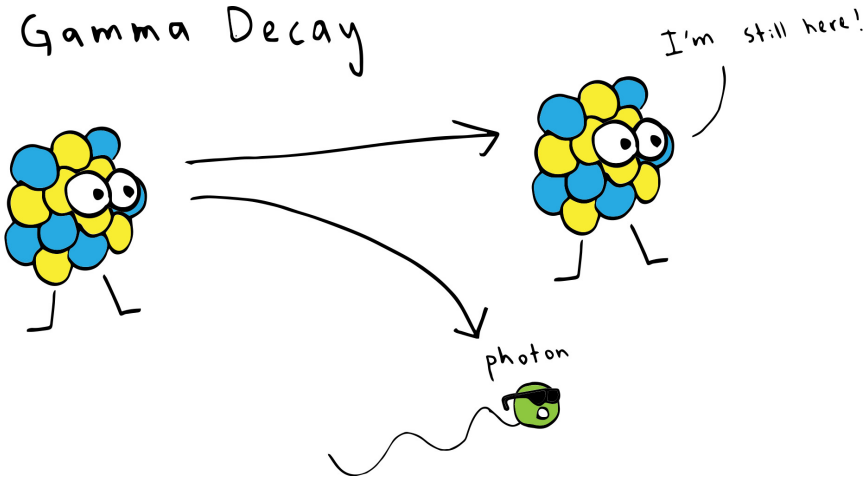
# Truth versus Provability






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So, a "proof system" or "proof calculus" typically specifies its axioms and also the accepted rules of deduction.

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|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
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# Gödel's Completeness Theorem

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Another way to think about this is: at the first-order level, every statement has a proof or a counterexample.

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Then $\Sigma \models Q$, but $\Sigma \nvdash Q$ for any proof system requiring finite-length proofs.

Gödel's Theorem does not apply in this setting, since $Q$ is not a first-order sentence.

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|---|---|---|---|---|
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# Examples!

Let's try these three forms of proof for a theorem concerning $\mathbb{R}$.

**Theorem.** If $0 < x < 1$, then $x^2 < x$.

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Assume that $0 < x < 1$. Since $0 < x$, multiplication by $x$ preserves inequalities. Multiply $x < 1$ by $x$ to obtain $x^2 < x$. $\square$

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